Individual Part

A (Algebra & Combinatorics).
Let X be a set with an operation on it $\diamond : X^2 \to X$ such that

(a) $x \diamond x = x$ for all $x \in X$,

(b) $x \diamond (y \diamond z) = z \diamond (x \diamond y)$ for all $x, y, z \in X$

Show that $\diamond$ is associative and commutative.

C (Calculus, Mathematical Analysis & Topology).
Let P be a polynomial of degree n with real coefficients such that $P(x) > 0$ for all real x. Show that

$$\forall_{x\in\mathbb{R}}\ P(x) + P'(x) + P''(x) + \cdots + P^{(n)}(x) > 0\,.$$

G (Linear Algebra & Geometry).
Let $A_n = [a_{j,k}] \in M_{n\times n}(\mathbb{R})$ be an $n\times n$ matrix with

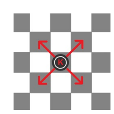
$$a_{j,k} = \big((j-1)n + k\big)^2.$$

for all $1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant n$. Find the rank of A_n

E (Equations & Inequalities).
Find all locally integrable functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the functional equation

$$\forall_{x\in\mathbb{R}_+}\ f(x) = \int_0^1 f(tx)\,dt\,.$$

P (Probability & Measure Theory).
In English draughts (also called American checkers) the pawn can only move diagonally to the unoccupied square. The king has the ability to move in all four diagonal directions, but still only one square.



Calculate the probability that the king will return to its starting position after $2n$ moves on the infinite checkerboard.

Team Part

A1 (Algebra & Combinatorics).
Let R be a ring such that $x^6 = x$ for all $x \in R$. Prove that R is commutative.

A2 (Algebra & Combinatorics).
Let $(A, +, \cdot)$ be a finite commutative ring where $a^2 = a$ for every element $a \in A$. Show that there exists a set X such that A is isomorphic to the ring ${\mathbb{F}_2}^X$, ie. to the ring of all functions on the set X with values in the smallest field of characteristics 2.
Is it also true for infinite rings A?

C1 (Calculus, Mathematical Analysis & Topology).
A closed subset E of a metric space (X, d) is called a Katowice set if every point of E is its accumulation point (an element a of a set A is called accumulation point if $\forall_\varepsilon \, \exists_{b\in A\setminus\{a\}} \, \mathrm{d}(a,b)<\varepsilon$). Prove that if E is an unbounded Katowice set, then it contains infinitely many bounded Katowice subsets.

C2 (Calculus, Mathematical Analysis & Topology).
Let the sequence (x_n) be defined as follows:

(i) $x_0 = 2$

(ii) $x_{n+1} = \dfrac{1}{x_n + \frac{1}{n+1}}$ for $n = 0, 1, 2, \dots$.

Prove that the sequence (x_n) converges.

G1 (Linear Algebra & Geometry).
Let $A, B \in M_{n\times n}(\mathbb{R})$ be two matrices such that $A^2 = A$, $B^2 = B$ and $\det(A + 3B) = 0$. Find $\det(3A + B)$.

G2 (Linear Algebra & Geometry).
Let $n > 1$ and $-\frac{1}{n-1} \leq c < 1$. Show, that there exist n vectors in $\mathbb{R}^n$ such that the angle between any two of them has cosine equal to c.

E1 (Equations & Inequalities).
Prove that

$$(1+\frac{1}{\sqrt{1}})(1+\frac{1}{\sqrt{2}})\cdot\ldots\cdot(1+\frac{1}{\sqrt{n}}) \geqslant \left(1+\frac{1}{\sqrt[2n]{n+1}}\sqrt{\frac{e}{n+1}}\right)^n$$

holds for every n.

E2 (Equations & Inequalities).
Find all functions $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ satisfying the following conditions:

1. $F(x, y) = F\left(\dfrac{x+y}{2}, \dfrac{2}{\frac{1}{x}+\frac{1}{y}}\right)$ for all $x, y > 0$.
2. $\frac{x+y}{2} \geq F(x, y) \geq \frac{2}{\frac{1}{x}+\frac{1}{y}}$ for all $x, y > 0$.

P1 (Probability & Measure Theory).
A ballot box contains $N+1$ balls numbered from 0 to N. We draw one ball from the box, then we throw away all balls with larger numbers and we return the drawn ball back to the ballot box. We repeat the draw with remaining balls.
Let x_n denotes the number of the ball drawn in n-th draw. Show that $\sum_{n=1}^{\infty} x_n < +\infty$ almost surely.

P2 (Probability & Measure Theory).
A team of three players participates in a game for a big prize. The host of the game-show places a hat on the head of each of the players. The hat is either white or red. The choice of the colours is random and their placements are independent. Each player can see the colours of the hats of his/her team mates but not his/her own.
The host asks the team players to make a guess at the same predetermined time. Each player can guess the colour of his/her hat, red or white, or can stay silent, i.e. pass. The team wins if at least one player guesses and all of those who guess do so correctly.
Find the strategy that maximises team chance of winning. What is the winning probability?

Solutions

Problem A.
Let X be a set with an operation on it $\diamond : X^2 \to X$ such that

(a) $x \diamond x = x$ for all $x \in X$,

(b) $x \diamond (y \diamond z) = z \diamond (x \diamond y)$ for all $x, y, z \in X$

Show that $\diamond$ is associative and commutative.

Solution:
First we show that $\diamond$ is commutative. We have

$$x \diamond y \overset{(a)}{=} (x \diamond y) \diamond (x \diamond y) \overset{2\times(b)}{=} x \diamond \big(y \diamond (x \diamond y)\big) .$$

Now

$$y \diamond (x \diamond y) \overset{2\times(b)}{=} x \diamond (y \diamond y) ,$$

so

$$x \diamond y = \dots = x \diamond \big(x \diamond (y \diamond y)\big) \overset{(b)}{=} (y \diamond y) \diamond (x \diamond x) \overset{(a)}{=} y \diamond x .$$

With commutativity we have

$$x \diamond (y \diamond z) \overset{(b)}{=} z \diamond (x \diamond y) \overset{(comm)}{=} (x \diamond y) \diamond z ,$$

which shows associativity. □

Problem A1.
Let R be a ring such that $x^6 = x$ for all $x \in R$. Prove that R is commutative.

Solution:
Since[1] $-x = (-x)^6 = x^6 = x$ we have $x + x = x - x = 0$, so $\operatorname{char} R = 2$. Now we have

$$x^2 + x = (x^2 + x)^6 = x^{12} + 6x^{11} + 15x^{10} + 20x^9 + 15x^8 + 6x^7 + x^6$$

$$= x^{12} + x^{10} + x^8 + X^6 = x^2 + x^4x + x^2x + x = x^5 + x^3 + x^2 + x,$$

hence $x^3 + x^5 = 0$, so $x^3 = x^5$. Now multiplying by x gives $x^4 = x^6 = x$ so $x^5 = x^2$, hence $x^3 = x^2$ so $x^4 = x^3$, $x^5 = x^4$, $x^6 = x^5$. Combining all equalities together gives $x^2 = x^3 = \dots = x^6 = x$ for all $x \in R$.
Thus $x + y = (x + y)^2 = x^2 + xy + yx + y^2 = x + xy + yx + y$, hence $xy + yx = 0$ so $xy = yx$ for all $x, y \in R$. □

Problem A2.
Let $(A, +, \cdot)$ be a finite commutative ring where $a^2 = a$ for every element $a \in A$. Show that there exists a set X such that A is isomorphic to the ring ${\mathbb{F}_2}^X$, ie. to the ring of all functions on the set X with values in the smallest field of characteristics 2.
Is it also true for infinite rings A?

Solution:
First let us note that (i) if $x \neq 0$ and $xy = 0$, then $x(x + y) = x^2 + xy = x \neq 0$ hence $x + y \neq 0$ for all $x, y \in A$; and (ii) as $x + y = (x + y)^2 = x^2 + xy + xy + y^2$ we have $xy + xy = 0$, so also $x^2 + x^2 = x + x = 0$ for all $x \in A$.
Now we will prove that the ring A is generated by its finite subset $X = \{s_1, \dots, s_n\}$ satisfying $ab = 0$ for all $a \neq b$, $a, b \in X$.
The set S we will construct by induction. If $A = \{0\}$, then $X = \varnothing$ and the isomorphism $A \sim {\mathbb{F}_2}^X$ is trivial.
On the other hand there is a non-zero element $a \in A$. The set $S_1 = \{0, a\}$ is obviously closed on both ring operations. Moreover this subring S_1 is generated by itself, so we put $X_1 = S_1$
Now let $X_k = \{x_1, \dots, x_k\}$ be the set of "orthogonal" generators of the subring S_k, ie. $xy = 0$ for all distict x and y in X_k. If $A \setminus S_k$ is not empty, there is a non-zero element $a \in A \setminus S_k$. Without loss of generality we can assume that $x_i a = 0$ for $i = 1, \dots, j$, and $x_i a \neq 0$ for $i = j + 1, \dots, k$.

[1] Since $x + (-x) = 0$ we have $0 = (-x)\big(x + (-x)\big) = (-x)x + (-x)^2$ and $0 = \big(x + (-x)\big)x = x^2 + (-x)x$, hence $(-x)^2 = x^2$.

Let now $X_{k+1} = \{x_1, \dots, x_j, ax_{j+1}, \dots, ax_k, x_{j+1} + \dots + x_k + a\}$ and let S_{k+1} be the subring of A generated by X_{k+1}. The last element of X_{k+1} is different from zero. Indeed, as $a \notin S_k$ we have $a \neq x_{j+1} + \dots + x_k$, hence $0 = a + a \neq x_{j+1} + \dots + x_k + a$ by (*ii*). Moreover all elements of X_{k+1} are mutually "orthogonal". To show this it is enough to see that

$$\begin{aligned} ax_l(x_{j+1} + \dots + x_k + a) &= ax_lx_{j+1} + \dots + ax_lx_k + ax_la \\ &= ax_l^2 + a^2x_l = ax_l + ax_l = 0 \text{ for } j < l \leq k\,. \end{aligned}$$

In this way we have constructed two sequences $X_1, X_2, \dots$ and $S_1, S_2, \dots$ of subsets of the ring A. Of course $|X_k| = k$, so $|S_k| = 2^k$ (in virtue of (*ii*) the elements of S_k, ie. of the set generated by X_k, are finite sums of elements of X_k). As the ring A is finite, there is an $N \in \mathbb{N}$ such that $S_N = A$. But all elements of S_N can be represented in the unique way as sums of elements of X_N:

$$S_n = \left\{ \sum_{k=1}^{N} \alpha_k x_k, \text{ for } \alpha_k \in \{0,1\}, x_k \in X_N \right\},$$

e.i. S_N is a linear space over the field $\mathbb{F}_2$. The correspondence

$$A \ni \sum_{k=1}^{N} \alpha_k x_k = x \mapsto f_x : X_N \to \mathbb{F}_2,\ f_x(x_k) = \alpha_k$$

is not only the isomorphism of the additive group structures of A and ${\mathbb{F}_2}^{X_N}$, but it extends also to the isomorphism of their multiplicative structures, as $x_ix_j = 0$ for $i \neq j$ and $x_ix_i = x_i$.

The answer to the second question is negative. It is enough to consider the ring of fuctions $\mathbb{N} \to \{0,1\}$ with finite support. Such a ring is countable. But the set ${\mathbb{F}_2}^X$ is either finite, or uncountable. □

Problem C.

Let P be a polynomial of degree n with real coefficients such that $P(x) > 0$ for all real x. Show that

$$\forall_{x\in\mathbb{R}}\ P(x) + P'(x) + P''(x) + \dots + P^{(n)}(x) > 0\,.$$

Solution:

Let us consider the function $g(x) = e^{-x}\big(P(x) + P'(x) + \dots + P^{(n)}(x)\big)$. Function g is differentiable infinitely many times. Moreover $\lim\limits_{x\to+\infty} g(x) = 0$. We have

$$\begin{aligned} g'(x) &= -e^{-x}\big(P(x) + P'(x) + \dots + P^{(n)}(x)\big) + e^{-x}\big(P'(x) + P''(x) + \dots + P^{(n+1)}(x)\big) \\ &= -e^{-x} f(x) \end{aligned}$$

as $P^{(n+1)}(x) \equiv 0$. So $g'(x) < 0$ for all $x \in \mathbb{R}$, thus the function g is decreasing to 0 on the whole real line. Hence it has to be positive everywhere. Finaly $P(x) = e^x g(x) > 0$ for all $x \in \mathbb{R}$. □

Problem C1.

A closed subset E of a metric space (X, d) is called a Katowice set if every point of E is its accumulation point (an element a of a set A is called accumulation point if $\forall_\varepsilon\, \exists_{b\in A\setminus\{a\}}\ \mathrm{d}(a,b) < \varepsilon$). Prove that if E is an unbounded Katowice set, then it contains infinitely many bounded Katowice subsets.

Solution:

At first we will show that if A is an open subset of X, the $\overline{E \cap A}$ is a Katowice set. Indeed, since $\overline{E \cap A}$ is a closed set, it's enough to show that every point of $\overline{E \cap A}$ is its accumulation point.

Assume conversely that there exists an element $x \in \overline{E \cap A}$ such that $x \notin (E \cap A)'$. It means that there exists $r_0 > 0$ such that

$$B_{r_0}(x) \cap \left(\overline{E \cap A}\right) = \{x\}$$

$\big($where $B_r(p) = \big\{y \in X : \mathrm{d}(y,p) < r\big\}$ is an open ball in the metric space $(X, \mathrm{d})\big)$. Also $x \in \overline{E \cap A}$ provides

$$B_{r_0}(x) \cap \big(E \cap A\big) \neq \varnothing,$$

so

$$B_{r_0}(x) \cap (E \cap A) = E \cap (B_{r_0}(x) \cap A) = \{x\}.$$

Since A is an open set, there exists $r_1 > 0$ such that

$$B_{r_1}(x) \subset B_{r_0}(x) \cap A,$$

so

$$\big(E \cap B_{r_1}(x)\big) \setminus \{x\} = \varnothing,$$

which contradicts the fact that $x \in E$ is an accumulation point of E. Hence $\overline{E \cap A}$ is a Katowice set.

Now we will construct an infinite family of bounded Katowice subsets of E. Let us take any $x_0 \in E$. The set $E_1 = \overline{E \cap B_1(x_0)}$ is then a Katowice subset of E and it is bounded as

$$E_1 \subset \overline{B_1(x_0)} = \big\{x \in X : \mathrm{d}(x, x_0) \leq 1\big\}.$$

There exists $x_1 \in E \setminus E_1$, because E is unbounded. So let $E_2 = \overline{E \cap B_{\mathrm{d}(x_1,x_0)+1}(x_0)}$. Of course $E_2 \ni x_1 \notin E_1$, so $E_1 \neq E_2$.
Now we can take $x_n \in E \backslash E_n$ for every $n \in \mathbb{N}$ and $E_{n+1} = \overline{E \cap B_{\mathrm{d}(x_n,x_0)+1}(x)}$. Sets $E_1, E_2, \ldots$ form an increasing, infinite sequence of different bounded Katowice subsets of E. □

Problem C2.
Let the sequence (x_n) be defined as follows:

(i) $x_0 = 2$

(ii) $x_{n+1} = \dfrac{1}{x_n + \frac{1}{n+1}}$ for $n = 0, 1, 2, \ldots$.

Prove that the sequence (x_n) converges.

Solution:
We have

$$\frac{1}{x_{n+2}} = \frac{1}{x_n + \frac{1}{n+1}} + \frac{1}{n+2} \tag{1}$$

We will show the inequality

$$x_{2n+1} < 1 - \frac{1}{2n+2} \tag{2}$$

using the mathematical induction on n.
It is easy to check (2) for $n = 0$. Let's assume it now for $n = k$. Equality (1) implies

$$\frac{1}{x_{2k+3}} = \frac{1}{x_{2k+1} + \frac{1}{2k+2}} + \frac{1}{2k+3} > \frac{1}{1 - \frac{1}{2k+2} + \frac{1}{2k+2}} + \frac{1}{2k+3} = \frac{2k+4}{2k+3},$$

hence

$$x_{2k+3} < \frac{2k+3}{2k+4} = 1 - \frac{1}{2(k+1)+2}\,.$$

Thus $x_{2n+1} < 1$ and

$$x_{2n+2} = \frac{1}{x_{2n+1} + \frac{1}{2n+2}} > \frac{1}{1 - \frac{1}{2n+2} + \frac{1}{2n+2}} = 1\,. \tag{3}$$

We will show now that the sequence (x_{2n}) is decreasing and the sequence (x_{2n+1}) is increasing. From the equation (1) we have

$$\frac{1}{n+2} = \frac{1}{x_{n+2}} - \frac{1}{x_n + \frac{1}{n+1}} = \frac{x_n - x_{n+2} + \frac{1}{n+1}}{x_{n+1}\big(x_n + \frac{1}{n+1}\big)}\,,$$

hence

$$(n+2)(x_n - x_{n+2}) = x_{n+1}\Big(x_n + \frac{1}{n+1}\Big) - \frac{n+2}{n+1} = (x_n x_{n+2} - 1) + \frac{x_{n+2}-1}{n+1}\,. \tag{4}$$

So we have $x_{2n} - x_{2n+2} > 0$ as $x_{2n}, x_{2n+2} > 1$, and $x_{2n+1} - x_{2n+3} < 0$ as $x_{2n+1}, x_{2n+3} < 1$. Hence sequences (x_{2n}) and (x_{2n+1}) are bounded and monotonic, so they converge. It remains to show that $x_{2n} - x_{2n+1}$ tends to 0. Indeed, from the equation (2) we have $\frac{1}{x_{2n+1}} > \frac{2n+2}{2n+1}$, so

$$x_{2n+1}x_{2n+2} = \frac{x_{2n+1}}{x_{2n+1} + \frac{1}{2n+2}} = \frac{2n+2}{2n+2+\frac{1}{x_{2n+1}}} < \frac{2n+2}{2n+2+\frac{2n+2}{2n+1}} = \frac{1}{1+\frac{1}{2n+1}} = \frac{2n+1}{2n+2}\,.$$

On the other hand

$$x_{2n}x_{2n+1} = \frac{x_{2n}}{x_{2n} + \frac{1}{2n+1}} < 1\,.$$

We have also

$$x_{n+1} - x_n = x_n x_{n+1}\Big(\frac{1}{x_n} - \frac{1}{x_{n+1}}\Big) = x_n x_{n+1}\Big(x_{n-1} + \frac{1}{n} - x_n - \frac{1}{n+1}\Big) = x_n x_{n+1}\Big(x_{n-1} - x_n + \frac{1}{n(n+1)}\Big)\,,$$

so

$$0 < x_{2n} - x_{2n+1} = x_{2n}x_{2n+1}\Big(x_{2n} - x_{2n-1} - \frac{1}{2n(2n+1)}\Big) < x_{2n} - x_{2n-1} - \frac{1}{2n(2n+1)}$$

and

$$\begin{aligned} 0 < x_{2n+2} - x_{2n+1} &= x_{2n+2}x_{2n+1}\Big(x_{2n} - x_{2n+1} + \frac{1}{(2n+1)(2n+2)}\Big) \\ &< \frac{2n+1}{2n+2}\Big(x_{2n} - x_{2n-1} - \frac{1}{2n(2n+1)} + \frac{1}{(2n+1)(2n+2)}\Big) < \frac{2n+1}{2n+2}\big(x_{2n} - x_{2n-1}\big). \end{aligned}$$

To finish the solution it is enough to see that

$$\prod_{k=0}^{n} \frac{2k+1}{2k+2} = \frac{1}{\prod\limits_{k=0}^{n}\big(1+\frac{1}{2k+1}\big)} < \frac{1}{1+\sum\limits_{k=0}^{n}\frac{1}{2k+1}} \overset{n\to\infty}{\longrightarrow} 0\,.$$

□

Problem G.
Let $A_n = [a_{j,k}] \in M_{n\times n}(\mathbb{R})$ be an $n\times n$ matrix with

$$a_{j,k} = \big((j-1)n+k\big)^2.$$

for all $1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant n$. Find the rank of A_n

Solution:
Suppose that $n \geqslant 3$. Let B_j be the j-th row of A_n . Then

$$r(A_n) = r\begin{pmatrix} B_1 \\ B_2 - B_1 \\ \vdots \\ B_n - B_{n-1} \end{pmatrix} = r\begin{pmatrix} B_1 \\ B_2 - B_1 \\ B_3 - 2B_2 + B_1 \\ \vdots \\ B_n - 2B_{n-1} + B_{n-2} \end{pmatrix}$$

and

$$\begin{gathered} a_{j,k} - a_{j-1,k} = ((j-1)n+k)^2 - ((j-2)n+k)^2 \\ = (j-1)^2n^2 - (j-2)^2n^2 + 2(j-1)nk - 2(j-2)nk + k^2 - k^2 = n((2j-3)n+2k) \end{gathered}$$

and

$$a_{j,k} - 2a_{j-1,k} + a_{j-2,k} = n((2j-3)n+2k) - n((2j-5)n+2k) = 2n.$$

Hence

$$r(A_n) = r\begin{pmatrix} 1, & 2^2, & 3^2, & \dots & n^2 \\ n+2, & n+4, & n+6, & \dots & n+2n \\ 2n, & 2n, & 2n, & \dots & 2n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2n, & 2n, & 2n, & \dots & 2n \end{pmatrix} \leqslant 3.$$

It is easy to check that $r(A_1) = 1$, $r(A_2) = 2$ and $r(A_n) = 3$ for every $n \geqslant 3$. □

Problem G1.
Let $A, B \in M_{n\times n}(\mathbb{R})$ be two matrices such that $A^2 = A$, $B^2 = B$ and $\det(A + 3B) = 0$. Find $\det(3A + B)$.

Solution:
Let $v \in \ker(A + 3B)$. Then $Av = -3Bv$, hence $(3A + B)v = -8Bv$. We have

$$(3A + B)^2 x = (3A + B)(3A + B)x = -8(3A + B)Bx = -24ABx - 8Bx.$$

On the other hand

$$(3A + B)^2 v = (9A + 3AB + 3BA + B)v = 9Av + 3ABv + 3BAv + Bv.$$

Hence

$$-24ABv = 9Av + 3ABv + 3BAv + 9Bv.$$

So

$$-6ABv = 3Av + ABv + BAv + 3Bv = 2Av + ABv + BAv + (A + 3B)v = 2Av + ABv + BAv.$$

Finaly we have $-7ABv = 2Av + BAv = 3Av$.
On the other hand we have $Av = -3Bv$, hence multiplying on the left-hand-side gives $A^2 v = Ax = -3ABv$. This means that $Av = 0$ and $Bv = 0$, so $v \in \ker A$ and $v \in \ker B$. Hence $v \in \ker(3A + B)$, which means $\det(3A + B) = 0$. □

Problem G2.
Let $n > 1$ and $-\frac{1}{n-1} \leq c < 1$. Show, that there exist n vectors in $\mathbb{R}^n$ such that the angle between any two of them has cosine equal to c.

Solution:
Let (e_i) be the canonical base of $\mathbb{R}^n$ and for $\alpha \in \mathbb{R}$ define $v_i = e_i + \alpha \sum_{k=1}^n e_k$. Easy calculation shows that

$$v_i \cdot v_j = n\alpha^2 + 2\alpha + \delta_{i,j}.$$

Thus cosine is equal to

$$\cos(v_j, v_k) = \frac{v_j \cdot v_k}{\sqrt{v_j \cdot v_j}\,\sqrt{v_k \cdot v_k}} = \frac{n\alpha^2 + 2\alpha}{n\alpha^2 + 2\alpha + 1} = 1 - \frac{1}{n\alpha^2 + 2\alpha + 1}.$$

The function $f(\alpha) = n\alpha^2 + 2\alpha + 1$ has minimal value $\frac{n-1}{n}$ at $\alpha = -\frac{1}{n}$ and its limits in $\pm\infty$ equal to $+\infty$. Thus minimal value of $\cos(v_j, v_k)$ is $1 - \frac{1}{\frac{n-1}{n}} = -\frac{1}{n-1}$ and the limit of $\cos(v_j, v_k)$ is 1 as $\alpha \to \pm\infty$. □

Problem E.
Find all locally integrable functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the functional equation

$$\forall_{x\in\mathbb{R}_+}\ f(x) = \int_0^1 f(tx)\,dt\,.$$

Solution:
By substitution $tx \mapsto s$ we have $f(x) = \int\limits_0^x f(s)\,\frac{ds}{x}$, hence function $xf(x) = \int\limits_0^x f(s)ds$, as a function of the upper limit of integration, is continuous. So it is also the function f itself. Hence the function $xf(x)$ is differentiable, so it also f itself. And so on, we get $f \in C^\infty$. Moreover, $xf(x)$ tends to 0 as $x \to 0^+$.
Now we have (by integration by parts)

$$f'(x) = \frac{d}{dx}\int\limits_0^1 f(tx)\,dt = \int\limits_0^1 \frac{d}{dx}f(tx)\,dt = \int\limits_0^1 tf'(tx)\,dt = \Big[t\frac{f(tx)}{x}\Big]_{t=0}^{t=1} - \frac{1}{x}\int\limits_0^1 f(tx)\,dt$$

$$= \Big(\frac{f(x)}{x} - 0\Big) - \frac{1}{x}\,f(x) = 0\,.$$

Thus the function f has to be a constant function. It is not hard to verify that $f(x) = c$ satisfies the integral equation in question. □

Remark.
The solution presented above assumes $\mathbb{R}_+ = (0, +\infty)$. In the case when $0 \in \mathbb{R}_+$ putting any value for $f(0)$ does not change any formula in the solution, as the value of the integral $f(x) = \int\limits_0^x f(s)\,ds$ does not depend on $f(0)$.

Problem E1.
Prove that

$$(1+\frac{1}{\sqrt{1}})(1+\frac{1}{\sqrt{2}})\cdot\ldots\cdot(1+\frac{1}{\sqrt{n}}) \geqslant \left(1+\frac{1}{\sqrt[2n]{n+1}}\sqrt{\frac{e}{n+1}}\right)^n$$

holds for every n.

Solution:
First we prove
Lemma.
For every positive real numbers $a_1, a_2, \ldots, a_n$ we have

$$\big(1+\sqrt[n]{a_1\cdot a_2\cdot\ldots\cdot a_n}\big)^n \leqslant (1+a_1)(1+a_2)\cdot\ldots\cdot(1+a_n).$$

Proof: Put $a_i = e^{x_i}$ for $i = 1, 2, \ldots, n$. First consider the function $f(x) = \ln(1+e^x)$. Since

$$f''(x) = \frac{e^x}{(1+e^x)^2} > 0$$

for every $x \in \mathbb{R}$, f is convex. Therefore

$$\begin{aligned}\ln\big(1+\sqrt[n]{a_1\cdot a_2\cdot\ldots a_n}\big) &= f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)\\ &\leqslant \frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}\\ &= \frac{\ln(1+a_1)+\ln(1+a_2)+\cdots+\ln(1+a_n)}{n}\\ &= \ln\big(\sqrt[n]{(1+a_1)\cdot(1+a_2)\cdot\ldots\cdot(1+a_n)}\big).\end{aligned}$$

Hence

$$\big(1+\sqrt[n]{a_1\cdot a_2\cdot\ldots\cdot a_n}\big)^n \leqslant (1+a_1)(1+a_2)\cdot\ldots\cdot(1+a_n).$$

End of the proof.

To finish the estimation we need the well known inequality

$$n! < (n+1)\frac{(n+1)^n}{e^n}$$

that holds for every n. $\square$

Problem E2.

Find all functions $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ satisfying the following conditions:

1. $F(x,y) = F\left(\dfrac{x+y}{2}, \dfrac{2}{\frac{1}{x}+\frac{1}{y}}\right)$ for all $x, y > 0$.
2. $\frac{x+y}{2} \geq F(x,y) \geq \frac{2}{\frac{1}{x}+\frac{1}{y}}$ for all $x, y > 0$.

Solution:

First let us note that the second condition gives $F(x,x) = x$ for all $x \in \mathbb{R}_+$.
Let us now consider two sequences (a_n) and (b_n) defined as follows:

$$a_0 = x,\ b_0 = y,$$

$$a_{n+1} = \frac{a_n + b_n}{2},\ b_{n+1} = \frac{2}{\frac{1}{a_n}+\frac{1}{b_n}}\,.$$

It is obvious that $a_n \geq b_n$ for $n = 1, 2\ldots$ (it is the well-known inequality between arithmetic and harmonic means). Moreover

$$a_{n+1} = \frac{a_n + b_n}{2} \leq \frac{a_n + a_n}{2} = a_n$$

and

$$b_{n+1} = \frac{2}{\frac{1}{a_n}+\frac{1}{b_n}} \geq \frac{2}{\frac{1}{b_n}+\frac{1}{b_n}} = b_n\,.$$

Hence both sequences converges. Let $\lim\limits_{n\to\infty} a_n = \alpha$ and $\lim\limits_{n\to\infty} b_n = \beta$. Then taking the limit with $n \to \infty$ in $a_{n+1} = \frac{a_n+b_n}{2}$ gives $\alpha = \frac{1}{2}(\alpha+\beta)$, hence $\alpha = \beta$.
Now, from the first condition on F, we have

$$F(x,y) = F(a_0,b_0) = \cdots = F(a_n,b_n)\,,$$

hence the second condition gives

$$a_{n+1} = \frac{a_n+b_n}{2} \geq F(x,y) = F(a_n,b_n) \geq \frac{2}{\frac{1}{a_n}+\frac{1}{b_n}} = b_{n+1}\,.$$

Thus

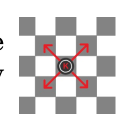
$$\alpha = \lim_{n\to\infty} a_n \geq F(x,y) \geq \lim_{n\to\infty} b_n = \beta = \alpha\,,$$

so $F(x,y) = \alpha$.
On the other hand

$$a_{n+1}b_{n+1} = \frac{a_n+b_n}{2}\,\frac{2}{\frac{1}{a_n}+\frac{1}{b_n}} = a_n b_n = \cdots = a_0 b_0 = xy\,,$$

hence $\alpha = \sqrt{xy}$. So the unique solution is the function $F(x,y) = \sqrt{xy}$. $\square$

Problem P.



In English draughts (also called American checkers) the pawn can only move diagonally to the unoccupied square. The king has the ability to move in all four diagonal directions, but still only one square.
Calculate the probability that the king will return to its starting position after $2n$ moves on the infinite checkerboard.

Solution:

The number of all paths of the king equals to 4^{2n}, hence the probability in question is the fraction of the number of all paths returning to the starting position over the number of all paths. Now, instead of paths on checkerboard we can consider paths on $\mathbb{Z}^2$ grid, by rotating the checkerboard clockwise by $\frac{\pi}{4}$. We can divide moves into two subsets: the set of horizontal moves and the set of vertical moves. The number of moves in both subsets has to be even, ie. the number of moves to the left has be equal to the number of moves to the right, the number of moves up has as well be equal to the number of moves down. So the number of all paths from $(0,0)$ to $(0,0)$ equals to the sum

$$\sum_{k=0}^{n}\binom{2n}{2k}\binom{2k}{k}\binom{2(n-k)}{n-k}.$$

Now we have

$$\binom{2n}{2k}\binom{2k}{k}\binom{2(n-k)}{n-k} = \frac{(2n)!}{(2k)!\,(2n-2k)!}\,\frac{(2k)!}{k!\,(n-k)!}\,\frac{(2n-2k)!}{(n-k)!\,k!}$$
$$= \frac{(2n)!}{n!\,n!}\,\frac{n!}{k!\,(n-k)!}\,\frac{n!}{(n-k)!\,k!} = \binom{2n}{n}\binom{n}{k}\binom{n}{n-k}.$$

On the other hand we know that $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$. So we have

$$\sum_{m}^{2n}\binom{2n}{m}a^m b^{2n-m} = (a+b)^{2n} = (a+b)^n\,(a+b)^n$$
$$= \left(\sum_{i=0}^{n}\binom{n}{k}a^k b^{n-k}\right)\left(\sum_{j=0}^{n}\binom{n}{j}a^j b^{n-j}\right)$$
$$= \sum_{i,j=0}^{n}\binom{n}{i}\binom{n}{j}a^{i+j}b^{n-i-j} = \sum_{m=0}^{2n}\sum_{i+j=m}\binom{n}{i}\binom{n}{j}a^m b^{2n-m}.$$

Now equality of the coefficients at $a^n b^n$ on both sides gives

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}.$$

So finaly the number of all paths is equal to

$$\sum_{k=0}^{n}\binom{2n}{2k}\binom{2k}{k}\binom{2(n-k)}{n-k} = \binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}^2,$$

and the probability in question is

$$\frac{\binom{2n}{n}^2}{4^{2n}}.$$

□

Solution 2:
As in the previous solutions, we have to calculate the number of closed paths of the lenght $2n$. Let us divide the set of moves which form one of such paths into four subsets $M_{Up,Right}$, $M_{Up,Left}$, $M_{Down,Right}$ and $M_{Down,Left}$, which represents moves in specific direction. If we consider only horizontal position of the king piece, its moves can be considered as moves on $\mathbb{Z}$-grid from 0 to 0, with corresponding subsets of moves H_{left} and H_{right}. The same happens when we consider vertical coordinate; in this case let us denote the corresponding subsets by V_{up} and V_{down}. The number of ways we can form subsets H_{right} and H_{left} (or equivalently subsets V_{up} and V_{down}) equals to the number of paths of length $2n$ starting from 0 on $\mathbb{Z}$-grid, which is equal to $\binom{2n}{n}$ (exactly half of the horizontal moves has to be in the left direction; the same is for vertical moves).
But we have a one-to-one correspondence between subsets H_{right}, H_{left}, V_{up}, V_{down} and subsets $M_{Up,Right}$, $M_{Up,Left}$, $M_{Down,Right}$, $M_{Down,Left}$, given by the formulas

$$H_{right} = M_{Up,Right} \cup M_{Down,Right}, \qquad H_{left} = M_{Up,Left} \cup M_{Down,Left},$$
$$V_{up} = M_{Up,Right} \cup M_{Up,Left}, \qquad V_{down} = M_{Down,Right} \cup M_{Down,Left},$$

and

$$M_{Up,Right} = V_{up} \cap H_{right}, \qquad M_{Up,Left} = V_{up} \cap H_{left},$$
$$M_{Down,Right} = V_{down} \cap H_{right}, \qquad M_{Down,Left} = V_{down} \cap H_{left}.$$

Because we can choose horizontal and vertical moves independently, the number of paths in question is equal to the product

$$\binom{2n}{n} \cdot \binom{2n}{n} = \binom{2n}{n}^2,$$

hence the probability equals to

$$\frac{\binom{2n}{n}^2}{4^{2n}}.$$

□

Remark.
As one can see from the result, the probability in question is the multiplication of probabilities of such moves on the line $\mathbb{Z}$. But it seems to be no evidence "a priori" of such independence.

Problem P1.
A ballot box contains $N+1$ balls numbered from 0 to N. We draw one ball from the box, then we throw away all balls with larger numbers and we return the drawn ball back to the ballot box. We repeat the draw with remaining balls.

Let x_n denotes the number of the ball drawn in n-th draw. Show that $\sum_{n=1}^{\infty} x_n < +\infty$ almost surely.

Solution:
We have the conditional probability

$$\mathbb{P}\big(x_{n+1} = k \mid x_n = m\big) = \frac{1}{m+1} = \frac{1}{x_n+1} \qquad \text{for } k = 0, 1, \dots, m,$$

so the conditional mean value equals to

$$\mathbb{E}\big(x_{n+1} \mid x_n\big) = \sum_{k=0}^{x_n} k \frac{1}{x_n+1} = \frac{x_n}{2},$$

where $x_0 = N$.

Let now $S_n = \sum_{k=1}^{n} x_k$ and $S = \sum_{n=1}^{\infty} x_n$. We have

$$\mathbb{E}(S_n) = \sum_{k=1}^{n} \mathbb{E}(x_k) = \sum_{k=1}^{n} \mathbb{E}\Big(\mathbb{E}(x_k \mid x_{k-1})\Big) \sum_{k=1}^{n} \frac{1}{2}\mathbb{E}(x_{k-1}) = \frac{1}{2}\mathbb{E}(S_{n-1}) + \frac{N}{2}.$$

Hence, as $\mathbb{E}(S_1) = \mathbb{E}(x_1) = \frac{N}{2}$, we have

$$\mathbb{E}(S_n) = \sum_{k=1}^{n} \frac{N}{2^k} < N.$$

Of course all random variables $S_1, S_2, \dots$ and S are non-negative, so by Chebyshev's inequality

$$\mathbb{P}\big(S_n \geq M\big) = \frac{\mathbb{E}(S_n)}{M} < \frac{N}{M},$$

for every $M > 0$. Hence

$$\mathbb{P}(S \geq M) = \lim_{n\to\infty} \mathbb{P}\big(S_n \geq M\big) \leq \frac{N}{M}.$$

So finaly

$$\mathbb{P}(S < +\infty) = \lim_{M\to\infty} \mathbb{P}(S < M) = \lim_{M\to\infty} \big(1 - \mathbb{P}(S \geq M)\big) \geq \lim_{M\to\infty} \Big(1 - \frac{N}{M}\Big) = 1.$$

□

Solution 2:
It is clear that the series $\sum_{n=1}^{\infty} x_n$ converges iff there is an n such that $x_n = 0$. So the set of events $\Big\{ \sum_{n=1}^{\infty} x_n < +\infty \Big\}$ is complement to the set $T = \{x_n > 0 \text{ for all } n \in \mathbb{N}\}$. Let $T_k = \{x_n > 0 \text{ for all } n \in \mathbb{N}, \text{ and } N = k\}$. After first drawing we are in the situation like as in the starting position but with different N (i.e. $N = x_1$ exactly). New series of random variables differs from the original only by shifted index. So we have

$$\mathbb{P}(T_n) = \sum_{k=0}^{n} \mathbb{P}(x_1 = k)\, \mathbb{P}(T_k) = \frac{\sum_{k=0}^{n} \mathbb{P}(T_k)}{n+1}\,,$$

hence

$$\mathbb{P}(T_n) = \frac{\sum_{k=0}^{n-1} \mathbb{P}(T_k)}{n}\,.$$

Of course we have $x_1 = 0$ for $N = 0$, so $\mathbb{P}(T_0) = 0$. Hence $\mathbb{P}(T_n) = 0$ for all $n \in \mathbb{N}$ by simply induction on n. This gives

$$\mathbb{P}\Big(\sum_{n=1}^{\infty} x_n < +\infty\Big) = 1 - \mathbb{P}(T_N) = 1\,.$$

□

Problem P2.
A team of three players participates in a game for a big prize. The host of the game-show places a hat on the head of each of the players. The hat is either white or red. The choice of the colours is random and their placements are independent. Each player can see the colours of the hats of his/her team mates but not his/her own.
The host asks the team players to make a guess at the same predetermined time. Each player can guess the colour of his/her hat, red or white, or can stay silent, i.e. pass. The team wins if at least one player guesses and all of those who guess do so correctly.
Find the strategy that maximises team chance of winning. What is the winning probability?

Solution:
Every player has two possibilities: to stay silent, ie. to pass, or to guess. Denote by S_i and G_i the corresponding subsets of the sample set Ω for the i-th player. Of course $S_i \cap G_i = \varnothing$ and $S_i \cup G_i = \Omega$. Moreover, each player makes the guess based on the colours of the hats he or she sees, i.e. only on the colours of other players' hats. Hence when the player makes a guess, he or she guesses correctly in exactly a half elementary events. Let us denote the correct guess by G_i^+ and the wrong by G_i^-. Hence $G_i^+ \cap G_i^- = \varnothing$, $G_i^+ \cup G_i^- = G_i$ and $\mathbb{P}(G_i^+) = \mathbb{P}(G_i^-)$. The team loses with the probability $\mathbb{P}\big(G_1^- \cup G_2^- \cup G_3^- \cup (S_1 \cap S_2 \cap S_3)\big)$. Its winning probability is less or equal to $\mathbb{P}(G_1^+ \cup G_2^+ \cup G_3^+) \le \mathbb{P}(G_1^+) + \mathbb{P}(G_2^+) + \mathbb{P}(G_3^+)$ and does not exceed $\mathbb{P}\big(\Omega \setminus (G_1^- \cup G_2^- \cup G_3^-)\big) \le 1 - \mathbb{P}(G_i^-)$. If we mark player 1 as the one with the highest probability of winning then the maximal winning probability is $3\mathbb{P}(G_1^+) \le 1 - \mathbb{P}(G_1^+)$. Thus, the highest winning probability value a team can achieve is $3/4$.

Now we will show a strategy that realizes the maximum probability value. There are eight possibilities of hats distribution: in two of them all players have hats of the same colour, and in the other six, two players have one colour and the third player has the other colour. The players can guarantee that they win in the latter cases (ie. in 75% of the time) with the following strategy:

(i) any player who observes two hats of two different colours remains silent;

(ii) any player who observes two hats of the same colour guesses the opposite colour.

In the two cases when all three players have the same hat colour, they will all guess incorrectly. But in the other six cases, only one player will guess, and correctly. So the probability of winning is exactly $3/4$. □