# Individual Part

**Problem A.**
Let $P$ be a polynomial of degree less than $n$. Show that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}P(k)=0\,.$$

**Problem C.**
Let $f\colon\mathbb{R}\to\mathbb{R}$ be twice differentiable at $x=0$ (i.e. $f$ is differentiable in some neighbourhood of $0$, but $f'$ does not need to be differentiable – or even continuous – outside $x=0$), and let

$$g(x)=\begin{cases} f'(0), & \text{for } x=0\\ \dfrac{f(x)-f(0)}{x}, & \text{for } x\neq 0.\end{cases}$$

Show that function $g$ is differentiable at $x=0$ and calculate $g'(0)$.

**Problem E.**
Find all differentiable functions $f\colon\mathbb{R}\to\mathbb{R}$ satisfying both following equations for all $x,y\in\mathbb{R}$

(I) $f(x-y)-f(x+y)=2\,f(x+1)\,f(y+1)$;

(II) $f(x)^2+f(x+1)^2=1$.

**Problem G.**
Let $A$ be a complex matrix of dimension $d\times d$. Assume that there exists a positive integer $n\in\mathbb{N}$ such that

$$A^{2n}+A^n+2nI=0,$$

where $I$ is the identity matrix. Prove that $A$ is diagonalizable.

**Problem P.**
Let $n$ be a fixed positive integer. Alice plays the following game. She puts $m$ perfect, symmetric coins into the game. Then she tosses all of them. The bank collects all those with “tail”, and doubles the remaining coins (those with “head”). The process then repeats until either all the coins show “tail”, or Alice has made $n$ tosses. The final gain is the number of remaining coins (i.e. the doubled number of “heads” after the last toss) minus the number of coins Alice put into the game at the beginning.
Find the expected value of Alice’s gain.

# Team Part

**Problem A.1.**
Let $G$ be a group with center $\mathrm{Z}(G)$. Prove that $\left| H/_{\mathrm{Z}(H)} \right| \leqslant \left| G/_{\mathrm{Z}(G)} \right|$ for every subgroup $H < G$.

Show moreover that if $G/_{\mathrm{Z}(G)}$ is finite then the equality holds iff $G = H\mathrm{Z}(G)$.

**Problem A.2.**
Billy the beetle walks along the edges of the cube. When it reaches a vertex, it continues along a randomly chosen one of the other two edges emerging from that vertex.
Calculate the number of all distinct paths the beetle can take to return to the starting point for the first time in $2n$ steps.

**Problem C.1.**
Let $f_n \colon [0,1] \to \mathbb{R}$ be given by the formula

$$f_n(x) = \frac{1}{n+1}\left(1+\frac{x}{1}\right)\cdot\left(1+\frac{x}{2}\right)\cdot\ldots\cdot\left(1+\frac{x}{n}\right)$$

for all $n \in \mathbb{N}$ and $x \in [0,1]$. Show that the sequence $(f_n)$ converges uniformly on the interval $[0,x]$ for each $0 < x < 1$. Find the pointwise limit of the sequence of functions.

**Problem C.2.**

Calculate the value of the integral $\displaystyle\int_0^\infty e^{-x^2}\cos x^2 \, \mathrm{d}x\,.$

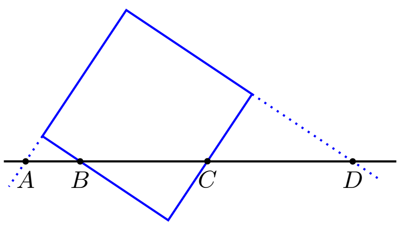
**Problem E.1.**
Let $n \in \mathbb{N}$, $n \geqslant 2$ and $a_1, \ldots, a_n$ be positive real numbers such that $a_1\, a_2 \,\cdots\, a_n = 1$. Show that

$$\sum_{k=1}^{n} \frac{\ln(a_k)}{1+a_k^2} \leqslant 0.$$

**Problem E.2.**
Let $n \geqslant 2$ be an integer and $x_1, \ldots, x_n$ distinct real numbers. Express in the simplest form the sum

$$\sum_{i=1}^{n} \frac{x_i^n}{\prod\limits_{j\neq i}(x_i - x_j)}\,.$$

**Problem G.1.**
Let $A, B, C, D$ be four different points on a straight line, in that order. Show that there exists a square such that two of these points lie on the sides of the square, and two other points lie on the extensions of two other sides of the square (see figure below).
Discuss the existence and number of different solutions.



**Problem G.2.**

Let $n \in \mathbb{N}$, $n > 1$. Let $X_n$ be the vector space generated by the set of all permutation matrices in the space $M_{n \times n}$ of all $n \times n$ real matrices. Find the dimension of the space $X_n$.

**Remark**

*For a given permutation $\sigma$ of the set $\{1, \ldots, n\}$, the permutation matrix $A_\sigma = [a_{k,m}] \in M_{n \times n}$ is defined by the formula*

$$a_{k,m} = \begin{cases} 1 & \textit{if } \sigma(k) = m; \\ 0 & \textit{if } \sigma(k) \neq m, \end{cases}$$

*for all $1 \leqslant k, m \leqslant n$.*

**Problem P.1.**

Find all $n \in \mathbb{N}$, $n \geqslant 2$ for which there exist Lebesgue measurable subsets $A_1, \ldots, A_n$ of $[0, 1]$ such that

(1) $\lambda(A_k) = \lambda(A_1) > 0$ for every $1 \leqslant k \leqslant n$ and

(2) $\sum_{k=1}^{n} \lambda(A_k) = \sum_{1 \leqslant j < k \leqslant n} \lambda(A_j \cap A_k)$

where $\lambda$ is the Lebesgue measure on $[0, 1]$.

**Problem P.2.**

In a volleyball match, each ball is served by one team and ends with one team winning one point. The next ball is served by the team which won the previous point. Typically, the probability $p$ of the serving team to win a point is different (smaller) than to lose a point. We assume that the results of each ball are independent. A set ends when one team has at least $25$ points and at least $2$ points more than the other team.

At some moment the score is $24:22$ for the team which won the last point. What is the expected final number of points of the team which will be victorious in this set?

# Solutions

**Problem A.**
Let $n$ be a positive integer and $P$ any polynomial of degree less than $n$. Show that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}P(k)=0\,.$$

**Solution 1.:**
We will show that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^j=0$$

for all $j=1,2,\dots,n-1$. The solution follows directly from this fact.
Let

$$f_0(x)=\sum_{k=0}^{n}\binom{n}{k}(-x)^k=(1-x)^n\,,$$

and let $f_k(x)=x\cdot f'_{k-1}(x)$ for $k=1,2,\dots,n$. It is easy to see that

$$f_k(x)=\sum_{j=0}^{n}j^k\binom{n}{j}(-x)^j\,,$$

for all $k\geq 1$. As $1$ is the zero of the multiplicity $n$ of the polynomial function $f_0$, the number $1$ is also a zero of every function $f_k$ for $k=1,2,\dots,n-1$. This means that

$$0=f_k(1)=\sum_{j=0}^{n}(-1)^j\binom{n}{j}j^k=0\,.$$

□

**Solution 2.:**
Let $\Delta$ be the difference operator $\Delta(f)(x)=f(x+1)-f(x)$. The operator $\Delta$ clearly reduces the degree of a polynomial by $1$, so $\Delta^n(f)=0$ if $f$ is a polynomial of degree less than $n$.
Now let $E$ be the shift operator $E(f)(x)=x+1$ and let $I$ denote the identity operator $I(f)=f$. Then we have

$$0=\Delta^n(f)(0)=(E-I)^n(f)(0)=\sum_k\binom{n}{k}(-1)^{n-k}E^k(f)(0)=\sum_k\binom{n}{k}(-1)^{n-k}f(k)$$

which is equivalent to the desired equation multiplied by $(-1)^n$. □

**Solution 3.:**
We will proceed by induction with respect to the degree of the polynomial $d=\deg P$.
Case $d=0$ means $P$ is constant, i.e. $P(x)=C$ for all $x$. Hence

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}P(k)=C\sum_{k=0}^{n}(-1)^k\binom{n}{k}=C(1-1)^n=0$$

for all $n\geq 1$.
Let us now assume that the equality holds for all polynomials of the degree less than $D$. Let now $d=\deg P=D\geq 1$. We have

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\binom{n}{k}P(k)&=\sum_{k=0}^{n}(-1)^k\left(\binom{n-1}{k-1}+\binom{n-1}{k}\right)P(k)\\&=\sum_{k=0}^{n-1}(-1)^{k+1}\binom{n-1}{k}P(k+1)+\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}P(k)\\&=\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\big(P(k)-P(k+1)\big)=0\end{aligned}$$

by the induction hypothesis, since the degree of the polynomial $Q(x)=P(x)-P(x+1)$ is equal to $d-1<D$. □

**Problem C.**
Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable at $x = 0$ (i.e. $f$ is differentiable on some neighbourhood of $0$, but $f'$ needs to be continuous – and differentiable – only at $0$), and let

$$g(x) = \begin{cases} f'(0), & x = 0 \\ \dfrac{f(x) - f(0)}{x}, & x \neq 0. \end{cases}$$

Show that function $g$ is differentiable at $x = 0$ and calculate $g'(0)$.

**Solution 1.:**
Let $\varepsilon > 0$ and let $\delta > 0$ be such that $\left|\dfrac{f'(x) - f'(0)}{x} - f''(0)\right| < \varepsilon$ for $|x| < \delta$.
Let now fix $x$ and let $x_k = \frac{k}{n}x$ for $k = 0, 1, \ldots, n$. We have $x_k - x_{k-1} = \frac{1}{n}x$, so

$$\frac{g(x) - g(0)}{x} = \frac{1}{x}\left(\frac{f(x) - f(0)}{x} - f'(0)\right) = \frac{1}{x}\left(\frac{\sum\limits_{k=1}^{n} f(x_k) - f(x_{k-1})}{x} - f'(0)\right)$$

$$= \frac{1}{x}\sum_{k=1}^{n}\frac{1}{n}\left(\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} - f'(0)\right) = \frac{1}{nx}\sum_{k=1}^{n}\big(f'(\xi_k) - f'(0)\big) = \frac{1}{n}\sum_{k=1}^{n}\frac{\xi_k}{x}\cdot\frac{f'(\xi_k) - f'(0)}{\xi_k}$$

for some $\xi_k$ in between $x_k$ and $x_{k-1}$, by Lagrange's Mean Value Theorem. It follows that $\dfrac{k-1}{n} = \dfrac{x_{k-1}}{x} < \dfrac{\xi_k}{x} < \dfrac{x_k}{x} = \dfrac{k}{n}$, hence

$$\frac{n(n-1)}{2n^2} = \frac{1}{n}\sum_{k=0}^{n-1}\frac{k}{n} < \frac{1}{n}\sum_{k=1}^{n}\frac{\xi_k}{x} < \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n} = \frac{n(n+1)}{2n^2}.$$

Thus the series $\frac{1}{n}\sum\limits_{k=1}^{n}\frac{\xi_k}{x}$ tends to $\frac{1}{2}$ as $n \to +\infty$.
Now if $|x| < \delta$ then

$$f''(0) - \varepsilon < \frac{f'(\xi) - f'(0)}{\xi} < f''(0) + \varepsilon$$

for all $|\xi| < |x|$. Hence

$$\frac{f''(0) - \varepsilon}{n}\sum_{k=1}^{n}\frac{\xi_k}{x} < \frac{g(x) - g(0)}{x} = \frac{1}{n}\sum_{k=1}^{n}\frac{\xi_k}{x}\cdot\frac{f'(\xi_k) - f'(0)}{\xi_k} < \frac{f''(0) + \varepsilon}{n}\sum_{k=1}^{n}\frac{\xi_k}{x}.$$

Passing with $n$ to infinity gives $\dfrac{1}{2}f''(0) - \dfrac{\varepsilon}{2} < \dfrac{g(x) - g(0)}{x} < \dfrac{1}{2}f''(0) + \dfrac{\varepsilon}{2}$, which shows that

$$\lim_{x \to 0}\frac{g(x) - g(0)}{x} = \frac{1}{2}f''(0).$$

**Remark**
*The solution can be written in a simpler form with uses of Landau's little $o$ symbol. We have* $\dfrac{f'(x) - f'(0)}{x} = f''(0) + o_x(1)$, *as* $x \to 0$. *Thus*

$$\frac{g(x) - g(0)}{x} = \cdots = \frac{1}{n}\sum_{k=1}^{n}\frac{\xi_k}{x}\cdot\frac{f'(\xi_k) - f'(0)}{\xi_k} = \frac{f''(0) + o_x(1)}{n}\sum_{k=1}^{n}\frac{\xi_k}{x} = \big(f''(0) + o_x(1)\big)\Big(\frac{1}{2} + o_n(1)\Big),$$

□

**Solution 2.:**
Let us expand the function $f$ in Taylor series at $x = 0$ with the remainder in Peano's form

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + o(x^2),$$

where $o$ is Landau's little "o" symbol as $x \to 0$. Thus $g(x) = f'(0) + \frac{f''(0)}{2}x + o(x)$ for $x \neq 0$, which gives

$$\frac{g(x) - g(0)}{x} = \frac{f''(0)}{2} + o(1) \overset{x \to 0}{\longrightarrow} \frac{f''(0)}{2}.$$

□

**Problem E.**
Find all differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the following equations for all $x, y \in \mathbb{R}$

(I) $f(x-y) - f(x+y) = 2\, f(x+1)\, f(y+1)$;

(II) $f(x)^2 + f(x+1)^2 = 1$.

**Solution:**
The solution are all functions of the form $\cos\left(\frac{2k+1}{2}\pi x\right)$ for $k \in \mathbb{Z}$. We proceed as follows. We can see from (II) that $f$ cannot be equal to $0$ everywhere. Putting now $y = 0$ gives

$$f(x) - f(x) = 0 = 2\, f(x+1)\, f(1)$$

for all $x$, hence $f(1) = 0$, and putting $x = 0$ leads to

$$f(-y) - f(y) = 2\, f(1)\, f(y+1) = 0$$

for all $y$, which shows that $f$ is even. Finaly substituting $y \to -y$ gives

$$f(x+y) - f(x-y) = 2\, f(x+1)\, f(-y+1) = 2\, f(x+1)\, f(y-1), \tag{22.1}$$

hence

$$2\, f(x+1)\, f(y-1) = -2\, f(x+1)\, f(y+1)$$

for all $x$ and $y$. This means that $f(y+1) = -f(y-1)$ for all $y \in \mathbb{R}$, or equivalently

$$f(x+2) = -f(x) \text{ for all } x \in \mathbb{R}. \tag{22.2}$$

Dividing (22.1) by $2y$ gives

$$\frac{f(x+y) - f(x-y)}{2y} = 2\, f(x+1)\, \frac{f(y-1)}{2y} = f(x+1)\, \frac{f(y-1) - f(-1)}{y},$$

as $f(-1) = f(1) = 0$, which tends to

$$f'(x) = f'(-1)\, f(x+1) \tag{22.3}$$

for all $x$ as $y \to 0$. This show that $f'$ is continuous and differentiable, hence $f$ is smooth (i.e. infinitely differentiable). Differetianting (22.3) and combining it with (22.2) gives the differential equation for $f$

$$f''(x) = -f'(-1)^2\, f(x),$$

which is known to have its only solutions of the form

$$f(x) = A\cos(\omega x) + B\sin(\omega x),$$

with $A, B \in \mathbb{R}$ and $\omega = \pm f'(-1)$.
We have $B = 0$ as $f$ is even, and $\omega = k\pi + \frac{\pi}{2} = \frac{2k+1}{2}\pi$ as $f(1) = 0$. Finaly we check that all functions $\cos\left(\frac{2k+1}{2}\pi x\right)$ for $k \in \mathbb{Z}$ satisfy both equations (I) and (II). $\square$

**Problem G.**

Let $A$ be a complex matrix of dimension $d \times d$. Assume that there exists a positive integer $n \in \mathbb{N}$ such that

$$A^{2n} + A^n + 2nI = 0,$$

where $I$ is the identity matrix. Prove that $A$ is diagonalizable.

**Solution:**

Let $\sigma(A)$ be a spectrum of $A$ and let $\lambda \in \sigma(A)$. Then there exists a nonzero vector $v_\lambda \in \mathbb{C}^d$ such that $Av_\lambda = \lambda v_\lambda$. Since $A^{2n} + A^n + 2nI = 0$, it follows that

$$(\lambda^{2n} + \lambda^n + 2n)v_\lambda = 0 \Longrightarrow \lambda^{2n} + \lambda^n + 2n = 0.$$

Let

$$p(x) = x^{2n} + x^n + 2n.$$

Then

$$p(A) = A^{2n} + A^n + 2nI = 0.$$

Moreover, one has

$$p'(x) = 2nx^{2n-1} + nx^{n-1} = nx^{n-1}(2x^n + 1).$$

Thus the polynomials $p(x)$ and $p'(x)$ do not have common roots, which implies that all the roots of $p(x)$ are distinct. Let $m_A(x)$ be the minimal polynomial of $A$. Since $p(A) = 0$, it follows that $m_A(x)$ divides the polynomial $p(x)$ (we are using the fact that the minimal polynomial of $A$ divides any polynomial $w(x)$ verifying the property $w(A) = 0$). Thus all roots of $m_A(x)$ are pairwise different. Finally, it suffices to use the fact that a matrix $A$ is diagonalizable if and only if the minimal polynomial $m_A(x)$ has no repeated roots. $\square$

**Problem P.**

Let $n$ be a fixed positive integer. Alice plays the following game. She puts $m$ perfect, symmetric coins into the game. Then she tosses all of them. The bank collects all those with “tail”, and doubles the remaining coins (those with “head”). The process then repeats until either all the coins show “tail”, or Alice has made $n$ tosses. The final gain is the number of remaining coins (i.e. the doubled number of “heads” after the last toss) minus the number of coins Alice put into the game at the beginning..

Find the expected value of Alice’s gain.

**Solution:**

Let $E_n$ be the expected value of the number of “heads” after the last toss if we start the game with one coin. The expected value of Alice’s winning then is equal to $m(2E_n - 1)$, as the bank independently pays out $2E_n$ coins for each one put into the game, from which the coin put in at the beginning must be subtracted.

If $n = 1$ the game ends after the first toss. So $E_1 = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}$. On the other hand, in the $n+1$-tosses game, after the first toss we either lose or continue with two coins in the $n$-tosses game. Hence we get the recursion $E_{n+1} = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 2E_n = E_n = \frac{1}{2}$. So the expected value of Alice’s winning is equal to

$$m(2E_n - 1) = m\Big(2 \cdot \frac{1}{2} - 1\Big) = 0\,.$$

$\square$

**Solution 2.:**

Let $G(m, n)$ denote Alice’s gain. Then of course $G(m, n) = mG(1, n)$. Moreover

$$G(1, n) = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot \big(1 + G(2, n-2)\big) = G(1, n-2) = \cdots = 0\,,$$

hence $G(m, n) = 0$. $\square$

**Problem A.1.**
Let $G$ be a group with center $\mathrm{Z}(G)$. Prove that $\left|H/_{\mathrm{Z}(H)}\right| \leqslant \left|G/_{\mathrm{Z}(G)}\right|$ for every subgroup $H < G$.

Show moreover that if $G/_{\mathrm{Z}(G)}$ is finite then the equality holds iff $G = H\mathrm{Z}(G)$.

**Solution 1.:**
Let $\mathrm{Inn}(G)$ denote the set of inner automorphisms of $G$, i.e.

$$\mathrm{Inn}(G) = \left\{\phi \,:\, \exists_{y\in G}\forall_{x\in G}\ \phi(x) = y^{-1}xy\right\}.$$

and let $\mathfrak{g} : G \to \mathrm{Inn}(G)$ be defined as $\mathfrak{g}(g)(x) = g^{-1}xg$. We have $g \in \ker\mathfrak{g}$ iff $\mathfrak{g}(g)(x) = x$, which is equivalent to $xg = gx$, for all $x \in G$. Hence $\ker\mathfrak{g} = \mathrm{Z}(G)$ and

$$\mathrm{Inn}(G) \simeq G/_{\mathrm{Z}(G)}\,, \tag{25.1}$$

by the first isomorphism theorem. The above formulas hold of course for any group.
Let now $H < G$, i.e. let $H$ be a subgroup of $G$. Any inner automorphism of the subgroup $H$ can be extended to an inner automorphism of the whole group $G$, so $\left|\,\mathrm{Inn}(H)\right| \leqslant \left|\,\mathrm{Inn}(G)\right|$. Hence the inequality in question follows from (25.1).
Let us now assume that $G/_{\mathrm{Z}(G)}$ is finite. Since $H\mathrm{Z}(G) < G$ and $H\cap \mathrm{Z}(G) < \mathrm{Z}(H)$ we have the following sequence

$$G/_{\mathrm{Z}(G)} > H\mathrm{Z}(G)/_{\mathrm{Z}(G)} \simeq H/_{H\cap \mathrm{Z}(G)} \xrightarrow{\mathfrak{f}} H/_{\mathrm{Z}(H)}\,,$$

where $\mathfrak{f}$ is epimorphism (i.e. surjective homomorphism) given by $\mathfrak{f}\big(h(H\cap \mathrm{Z}(G))\big) = h\mathrm{Z}(H)$ (it does not depend of the representative $h$). Hence if $\left|G/_{\mathrm{Z}(G)}\right| = \left|H/_{\mathrm{Z}(H)}\right|$ then $\left|G/_{\mathrm{Z}(G)}\right| = \left|H\mathrm{Z}(G)/_{\mathrm{Z}(G)}\right|$, so from the finiteness of the number of layers Z(G) follows the equality $G = H\mathrm{Z}(G)$.
On the other hand, if $G = H\mathrm{Z}(G)$, then for all $g \in G$ there exists an $h \in H$ such that $gh^{-1} \in \mathrm{Z}(G)$. Thus $gh^{-1}x = xgh^{-1}$, i.e. $g^{-1}xg = hxh^{-1}$ for all $x \in G$. This equality shows that every inner automorphism of $G$ can be uniquely determined by an inner automorphism of $H$, hence $\mathrm{Inn}(H) \simeq \mathrm{Inn}(G)$, which is equivalent to $H/_{\mathrm{Z}(H)} \simeq G/_{\mathrm{Z}(G)}$, so $\left|G/_{\mathrm{Z}(G)}\right| = \left|H/_{\mathrm{Z}(H)}\right|$. □

**Remark**
*The finitness of* $\left|G/_{\mathrm{Z}(G)}\right|$ *is needed only for the implication*

$$\left|G/_{\mathrm{Z}(G)}\right| = \left|H/_{\mathrm{Z}(H)}\right| \quad\Rightarrow\quad G = H\mathrm{Z}(G)\,.$$

**Solution 2.:**
Two elements in $H$ are equal in $G/_{\mathrm{Z}(G)}$ if there exists a $g$ in $\mathrm{Z}(G)$ such that $h_1g = h_2$, but this element $g$ is clearly an element of $H$ and as an element of $H$ and $\mathrm{Z}(G)$ clearly an element of $\mathrm{Z}(H)$. So $h_1$ and $h_2$ are already the same in $G/_{\mathrm{Z}(H)}$.
Therefore, we don't lose anything. The only way to have equality is the ability to choose a representative in $H$ for each class which is easily equivalent to the desired condition $G = H\mathrm{Z}(G)$.
□

**Problem A.2.**

Billy the beetle walks along the edge of the cube. When it reaches a vertex, it continues along a randomly chosen one of the other two edges emerging from that vertex.
Calculate the number of all the different paths the beetle can take to return to the starting point for the first time in $2n$ steps.

**Solution 1.:**

We assume that vertices of a unit cube have coordinates: $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(0,1,1)$, $(1,0,0)$, $(1,0,1)$, $(1,1,0)$ and $(1,1,1)$. Let us divide the vertices into four sets according to the sum of the coordinate values $S_0,\ldots,S_3$. So there are no direct connection between vertices in the same set, moreover sets $S_0$ and $S_3$ contain only one vertex (exactly the opposite vertices). Let now $P_n(i\to j)$ denote the number of paths of length $n$ starting at $(0,0,0)$ and in the last step going from a vertex in $S_i$ to a vertex in $S_j$. Thus, the value we are looking for is $c_n = P_{2n}(1\to 0)$. Moreover we can assume that the beetle stops when it reaches the starting point before $2n$ steps, so we discard these paths since we only count those in which it returns for the first time in exactly $2n$ steps. Hence $P_n(0\to 1) = 0$ for all $n \geq 2$, therefore we take as the initial condition $P_2(1\to 2) = 6$.
Thus the system of equations describing the number of paths is as follows:

$$
\begin{aligned}
&P_{n+1}(1\to 0) = P_n(2\to 1), && P_2(1\to 0) = 0,\\
&P_{n+1}(1\to 2) = P_n(2\to 1), && P_2(1\to 2) = 6,\\
&P_{n+1}(2\to 1) = 2P_n(3\to 2) + P_n(1\to 2), && P_2(2\to 1) = 0,\\
&P_{n+1}(2\to 3) = P_n(1\to 2), && P_2(2\to 3) = 0,\\
&P_{n+1}(3\to 2) = 2P_n(2\to 3), && P_2(3\to 2) = 0.
\end{aligned}
$$

So

$$
\begin{aligned}
P_{n+2}(1\to 0) = P_{n+2}(1\to 2) = P_{n+1}(2\to 1) &= 2P_n(3\to 2) + P_n(1\to 2) = 4P_{n-1}(2\to 3) + P_n(1\to 2)\\
&= 4P_{n-2}(1\to 2) + P_n(1\to 2) = P_n(1\to 0) + 4P_{n-2}(1\to 0),
\end{aligned}
$$

which gives $c_{n+1} = c_n + 4c_{n-1}$ with the initial conditions $c_1 = 0$, $c_2 = 6$. The characteristic equation $0 = \lambda^2 - \lambda - 4$ has two solutions $\lambda_+ = \frac{1+\sqrt{17}}{2}$ and $\lambda_- = \frac{1-\sqrt{17}}{2}$. Hence puting $c_n = a_+\lambda_+^{n-1} + a_-\lambda_-^{n-1}$ into the initial conditions gives $a_+ = -a_- = \frac{6}{\sqrt{17}}$, so finaly

$$
P_{2n}(1\to 0) = c_n = \frac{6}{\sqrt{17}}\,\frac{(1+\sqrt{17})^{n-1} - (1-\sqrt{17})^{n-1}}{2^{n-1}}.
$$

□

**Solution 2.:**

Denote the vertices of a unit cube as in solution 1. Let us divide the vertices into two sets: $S_o$ and $S_\times$, with $S_o = \big\{(0,0,0),\ (1,1,1)\big\}$. So there are no direct connection between vertices inside $S_o$. Moreover the vertex $(0,0,0)$ can be reached only in even number of steps and the vertex $(1,1,1)$ only in odd number of steps. Hence our goal is to calculate all paths of the length $2n$, starting at $(0,0,0)$ and moving from $S_\times$ to $S_o$ in the last step.
Let $p_n(i\to j)$ denote the probability that a path of the length $n$ starting at $(0,0,0)$ ends with the move from $S_i$ to $S_j$. As the number of all paths of the length $n$ is equal to $3\cdot 2^{n-1}$ (at first step the beetle has three possibilities, and then on every other step only two), the number we are looking for is equal to $3\cdot 2^{2n-1}\cdot p_{2n}(\times\to\circ)$.
We have of course $p_n(\circ\to\circ) + p_n(\circ\to\times) + p_n(\times\to\times) + p_n(\times\to\circ) = 1$. We assume moreover that the beetle stops when it reaches the starting point. The assumptions of the problem give the initial probabilities $p_1(\circ\to\times) = p_2(\times\to\times) = 1$ and the following equations of transition

$$
\begin{aligned}
p_{n+1}(\times\to\circ) &= \frac{1}{2}p_n(\times\to\times),\\
p_{n+1}(\times\to\times) &= \frac{1}{2}p_n(\times\to\times) + p_n(\circ\to\times),\\
p_{2n}(\circ\to\times) &= p_{2n-1}(\times\to\circ) \quad \& \quad p_{2n+1}(\circ\to\times) = 0,\\
p_{2n}(\circ\to\circ) &= p_{2n-1}(\circ\to\circ) \quad \& \quad p_{2n+1}(\circ\to\circ) = p_{2n}(\circ\to\circ) + p_{2n}(\times\to\circ).
\end{aligned}
$$

Thus

$$\begin{aligned}
p_{2n+1}(\times\to\times) &= \frac{1}{2}p_{2n}(\times\to\times) + p_{2n}(\circ\to\times) = \frac{1}{2}p_{2n}(\times\to\times) + p_{2n-1}(\times\to\circ) \\
&= \frac{1}{2}p_{2n}(\times\to\times) + \frac{1}{2}p_{2n-2}(\times\to\times) \\
p_{2n+2}(\times\to\times) &= \frac{1}{2}p_{2n+1}(\times\to\times) + p_{2n+1}(\circ\to\times) = \frac{1}{2}p_{2n+1}(\times\to\times),
\end{aligned}$$

so

$$p_{2n+3}(\times\to\times) = \frac{1}{2}p_{2n+2}(\times\to\times) + \frac{1}{2}p_{2n}(\times\to\times) = \frac{1}{4}p_{2n+1}(\times\to\times) + \frac{1}{4}p_{2n-1}(\times\to\times)$$

and hence

$$p_{2n+4}(\times\to\circ) = \frac{1}{4}p_{2n+2}(\times\to\circ) + \frac{1}{4}p_{2n-1}(\times\to\circ)$$

Let $P_n = p_{2n}(\times\to\circ)$, so we have to solve $4P_{n+2} = P_{n+1} + P_n$ with the initial conditions $P_1 = 0$, $P_2 = \frac{1}{4}$. The characteristic equation $0 = 4\lambda^2 - \lambda - 1$ has two solutions $\lambda_+ = \frac{1+\sqrt{17}}{8}$ and $\lambda_- = \frac{1-\sqrt{17}}{8}$ and we proceed as in solution 1 to get $P_n = \dfrac{(1+\sqrt{17})^{n-1} - (1-\sqrt{17})^{n-1}}{8^{n-1}\sqrt{17}}$, which multiplied by $3\cdot 2^{2n-1}$ gives the desired result. □

**Problem C.1.**
Let $f_n\colon[0,1]\to\mathbb{R}$ be given by the formula

$$f_n(x)=\frac{1}{n+1}\Big(1+\frac{x}{1}\Big)\cdot\Big(1+\frac{x}{2}\Big)\cdot\ldots\cdot\Big(1+\frac{x}{n}\Big)$$

for all $n\in\mathbb{N}$ and $x\in[0,1]$. Show that the sequence $(f_n)$ converges uniformly on the interval $[0,x]$ for each $0<x<1$. Find the pointwise limit of the sequence of functions.

**Solution:**
By the AM-GM inequality

$$f_n(x)=\frac{1}{n+1}\Big(1+\frac{x}{1}\Big)\cdot\Big(1+\frac{x}{2}\Big)\cdot\ldots\cdot\Big(1+\frac{x}{n}\Big)\leqslant\frac{1}{n+1}\bigg(1+\frac{x(1+\frac{1}{2}+\ldots+\frac{1}{n})}{n}\bigg)^n$$

for every $0\leqslant x\leqslant 1$.
In the sequel we will need the folowing two well known facts
**Lemma**

$(a)$ *For all $n\in\mathbb{N}$ and $x\geqslant 0$, we have*

$$\Big(1+\frac{x}{n}\Big)^n\leqslant e^x.$$

$(b)$ *For every $n\in\mathbb{N}$, we have*

$$1+\frac{1}{2}+\ldots+\frac{1}{n}\leqslant 1+\ln(n).$$

Consequently

$$\begin{aligned}0\leqslant f_n(x)&\leqslant\frac{1}{n+1}\bigg(1+\frac{x(1+\frac{1}{2}+\ldots+\frac{1}{n})}{n}\bigg)^n\leqslant\frac{e^{x(1+\ln(n))}}{n+1}=e^x\bigg(\frac{n^x}{n+1}\bigg)\\&=e^x\bigg(\frac{n}{n+1}\bigg)^x\bigg(\frac{1}{n+1}\bigg)^{1-x}\end{aligned}$$

for every $0\leqslant x\leqslant 1$. Let $0<y<1$. Then

$$\sup_{0\leqslant x\leqslant y}|f_n(x)|\leqslant e^y\bigg(\frac{n}{n+1}\bigg)^0\bigg(\frac{1}{n+1}\bigg)^{1-y}\leqslant\frac{e^y}{(n+1)^{1-y}}.$$

This shows that

$$\lim_{n\to\infty}\sup_{0\leqslant x\leqslant y}|f_n(x)|=0.$$

It is easy to check that

$$\lim_{n\to\infty}f_n(1)=1.$$

Therefore

$$\lim_{n\to\infty}f_n(x)=\begin{cases}1&\text{if } x=1\\0&\text{if } 0\leqslant x<1\end{cases}$$

for every $0\leqslant x\leqslant 1$.

□

**Remark**
*] It's quicker to note immediately that $1+\dfrac{x}{k}<e^{x/k}$ since $x$ is positive and we have part of the Taylor series of* exp.

**Problem C.2.**

Calculate the value of the integral $\displaystyle\int_0^\infty \cos x^2\, e^{-x^2}\,\mathrm{d}x\,.$

**Solution:**
Let's denote the integral in question by $I$. Thus

$$
\begin{aligned}
I^2 &= \int_0^\infty\int_0^\infty \cos x^2\ \cos y^2\, e^{-x^2-y^2}\,\mathrm{d}x\mathrm{d}y = \int_0^\infty\int_0^\infty \frac{\cos(x^2+y^2)-\cos(x^2-y^2)}{2}\, e^{-(x^2+y^2)}\,\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}}\mathrm{d}t\int_0^\infty \big(\cos r^2+\cos(r^2\cos 2t)\big)\, e^{-r^2}\, r\mathrm{d}r = \frac{1}{4}\int_0^{\frac{\pi}{2}}\mathrm{d}t\int_0^\infty \big(\cos s+\cos(s\cos 2t)\big)\, e^{-s}\,\mathrm{d}s \\
&= \frac{1}{4}\int_0^{\frac{\pi}{2}}\Big(\frac{1}{2}+\frac{1}{1+\cos^2 2t}\Big)\,\mathrm{d}t = \frac{1}{4}\Big(\frac{\pi}{4}+\frac{\pi}{2\sqrt{2}}\Big) = \frac{1+\sqrt{2}}{16}\pi\,.
\end{aligned}
$$

where

$$
\begin{aligned}
\int_0^\infty \cos(ax)\, e^{-x}\,\mathrm{d}x &= -\cos(ax)e^{-x}\Big|_0^\infty - a\int_0^\infty \sin(ax)\, e^{-x}\,\mathrm{d}x \\
&= 1 + a\Big[\sin(ax)e^{-x}\Big]_0^\infty - a^2\int_0^\infty \cos(ax)e^{-x}\,\mathrm{d}x,
\end{aligned}
$$

hence $\displaystyle\int_0^\infty \cos(ax)\, e^{-x}\,\mathrm{d}x = \frac{1}{1+a^2}$, and

$$
\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos^2 2t}\,\mathrm{d}t = \int_0^\infty \frac{1}{x^2+2}\,\mathrm{d}x = \frac{\pi}{2\sqrt{2}}
$$

by substitution $x=\tan t$.
Finaly

$$
I = \int_0^\infty \cos x^2\, e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{1+\sqrt{2}}}{4}\sqrt{\pi}\,.
$$

□

**Solution 2.:**
We have

$$
\cos x = \sum_{n=0}^\infty (-1)^n\frac{x^{2n}}{(2n)!}
$$

hence

$$
\int_0^\infty \cos(x^2)\, e^{-x^2}\,\mathrm{d}x = \sum_{n=0}^\infty\int_0^\infty (-1)^n\frac{x^{4n}}{(2n)!}\, e^{-x^2}\,\mathrm{d}x\,.
$$

Now

$$\begin{aligned}
\frac{1}{(2n)!}\int_0^\infty x^{4n}\,e^{-x^2}\,\mathrm{d}x =& \frac{1}{2(2n)!}\int_0^\infty x^{4n-1}\,e^{-x^2}\,2x\mathrm{d}x\\
=& \frac{4n-1}{2(2n)!}\int_0^\infty x^{4n-2}\,e^{-x^2}\,\mathrm{d}x\\
\vdots&\\
=& \frac{\prod_{k=1}^{2n}(2k-1)}{2^{2n}(2n)!}\int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{(4n-1)!!}{(4n)!!}\sqrt{\frac{\pi}{2}}\,\mathrm{d}x\,.
\end{aligned}$$

On the other hand

$$\sqrt{1+x} = \sum_{n=0}^\infty (-1)^n \frac{(2n-1)!!}{(2n)!!}x^n$$

□

**Problem E.1.**
Let $n \in \mathbb{N}$, $n \geqslant 2$ and $a_1, \ldots, a_n$ be positive real numbers such that $a_1\, a_2\, \cdots\, a_n = 1$. Show that

$$\sum_{k=1}^{n} \frac{\ln(a_k)}{1+a_k^2} \leqslant 0.$$

**Solution:**
Without loss of generality we may assume that $a_1 \leqslant \ldots \leqslant a_n$. Then

$$\begin{gathered}\ln(a_1) \leqslant \ldots \leqslant \ln(a_n),\\ \frac{1}{1+a_1^2} \geqslant \ldots \geqslant \frac{1}{1+a_n^2},\\ \sum_{k=1}^{n} \ln(a_k) = 0,\\ \sum_{k=1}^{j} \ln(a_k) \leqslant 0\end{gathered}$$

for every $1 \leqslant j < n$. According to Abel's lemma we obtain

$$\begin{aligned}-\sum_{k=1}^{n} \frac{\ln(a_k)}{1+a_k^2} &= \frac{1}{1+a_1^2}(-\ln(a_1)) + \sum_{k=2}^{n} \frac{1}{1+a_k^2}\left(\sum_{j=1}^{k}(-\ln(a_j)) - \sum_{j=1}^{k-1}(-\ln(a_j))\right)\\ &= \sum_{k=1}^{n-1}\left(\frac{1}{1+a_k^2} - \frac{1}{1+a_{k+1}^2}\right)\left(\sum_{j=1}^{k}(-\ln(a_j))\right) \geqslant 0.\end{aligned}$$

Equality holds only for $a_1 = \cdots = a_n = 1$. □

**Solution 2.:**
We have $1+a^2 < 1+a^2b^2 < 1+b^2$ for $0 < a < 1 < b$, hence

$$\frac{\ln a}{1+a^2} + \frac{\ln b}{1+b^2} < \frac{\ln a}{1+a^2b^2} + \frac{\ln b}{1+a^2b^2} = \frac{\ln(ab)}{1+(ab)^2}\,.$$

Iterating gives immediately the desired result. □

**Problem E.2.**

Let $n \geqslant 2$ be an integer and $x_1, \ldots, x_n$ distinct real numbers. Express in the simplest form the sum

$$\sum_{i=1}^{n} \frac{x_i^n}{\prod\limits_{j \neq i} (x_i - x_j)} .$$

**Solution:**

The series is equal to

$$\sum_{i=1}^{n} \frac{x_i^n}{\prod\limits_{j \neq i} (x_i - x_j)} = x_1 + \cdots + x_n .$$

We will procede by induction with respect to $n$.

If $n = 2$ we have

$$\sum_{i=1}^{2} \frac{x_i^2}{\prod\limits_{j \neq i} (x_i - x_j)} = \frac{x_1^2}{(x_1 - x_2)} + \frac{x_2^2}{(x_2 - x_1)} = \frac{x_1^2 - x_2^2}{x_1 - x_2} = x_1 + x_2 .$$

Let now

$$f_n(x_1, \ldots, x_n) = \sum_{i=1}^{n} \frac{x_i^n}{\prod\limits_{j \neq i} (x_i - x_j)} ,$$

and assume that the equality $f_n(x_1, \ldots, x_n) = x_1 + \cdots + x_n$ holds for some $n \geqslant 2$.

We have

$$\begin{aligned}
\frac{x_i^{n+1}}{\prod\limits_{j \neq i} (x_i - x_j)} &= \frac{x_i^n}{\prod\limits_{2 \leqslant j \neq i}^{n} (x_i - x_j)} \frac{x_i}{(x_i - x_1)(x_i - x_{n+1})} \\
&= \frac{x_i^n}{\prod\limits_{2 \leqslant j \neq i}^{n} (x_i - x_j)} \frac{x_i(x_1 - x_{n+1})}{(x_i - x_1)(x_i - x_{n+1})(x_1 - x_{n+1})} \\
&= \frac{x_i^n}{\prod\limits_{2 \leqslant j \neq i}^{n} (x_i - x_j)} \frac{x_1(x_i - x_{n+1}) - x_{n+1}(x_i - x_1)}{(x_i - x_1)(x_i - x_{n+1})(x_1 - x_{n+1})} \\
&= \frac{x_i^n}{\prod\limits_{2 \leqslant j \neq i}^{n} (x_i - x_j)} \left( \frac{x_1}{(x_i - x_1)(x_1 - x_{n+1})} + \frac{x_{n+1}}{(x_i - x_{n+1})(x_{n+1} - x_1)} \right) \\
&= \frac{x_1}{x_1 - x_{n+1}} \frac{x_i^n}{\prod\limits_{1 \leqslant j \neq i}^{n} (x_i - x_j)} + \frac{x_{n+1}}{x_{n+1} - x_1} \frac{x_i^n}{\prod\limits_{2 \leqslant j \neq i}^{n+1} (x_i - x_j)} .
\end{aligned}$$

Hence

$$\begin{aligned}
f_{n+1}(x_1, \ldots, x_{n+1}) &= \frac{x_1^{n+1}}{\prod\limits_{j=2}^{n+1} (x_1 - x_j)} + \sum_{i=2}^{n} \frac{x_i^{n+1}}{\prod\limits_{j \neq i} (x_i - x_j)} + \frac{x_{n+1}^{n+1}}{\prod\limits_{j=1}^{n} (x_1 - x_j)} \\
&= \frac{x_1}{x_1 - x_{n+1}} \sum_{i=1}^{n} \frac{x_1^n}{\prod\limits_{1 \leqslant j \neq i}^{n} (x_i - x_j)} + \frac{x_{n+1}}{x_{n+1} - x_1} \sum_{i=2}^{n+1} \frac{x_{n+1}^n}{\prod\limits_{2 \leqslant j \neq i}^{n+1} (x_i - x_j)} \\
&= \frac{x_1}{x_1 - x_{n+1}} f_n(x_1, \ldots, x_n) + \frac{x_{n+1}}{x_{n+1} - x_1} f_n(x_2, \ldots, x_{n+1}) \\
&= \frac{x_1^2 + (x_1 - x_{n+1})(x_2 + \cdots + x_n) - x_{n+1}^2}{x_1 - x_{n+1}} \\
&= x_1 + \cdots + x_{n+1} ,
\end{aligned}$$

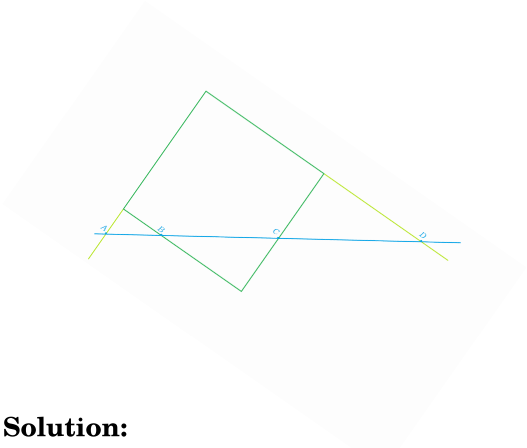
what finishes the proof. $\square$

**Remark**

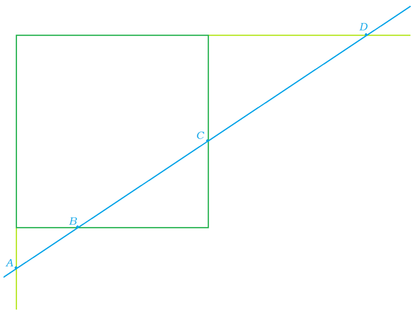
*The problem can also be solved by interpreting it as Lagrange interpolation.*

**Problem G.1.**

Let $A, B, C, D$ be four different points on a straight line, in that order. Show that there exists a square such that two of these points lie on the sides of the square, and two other points lie on the extensions of two other sides of the square (see figure below).
Discuss the existence and number of different solutions.




**Solution:**
First, let us note that no vertex of the square can coincide with points $A, B, C$ or $D$ (otherwise two of the points $A, B, C$ or $D$ should have coincided). Secondly, the points on the extensions of the sides must lie outside the square, so they cannot be between points that lie on the sides of the square. Thus, the points $A$ and $D$ must lie on the extensions of the sides, and the points $B$ and $C$ must lie on the sides of the square.
On the other hand, consider two pairs of parallel lines containing the sides of a possible square. That is, these pairs are determined either by the pairs $A, C$ and $B, D$, or by the pairs $A, D$ and $B, C$ (the points $B$ and $A$ lie, respectively, on the side and on the extension of the next side perpendicular to the first; so they cannot belong to the pair of parallel lines in question).




Without loss of generality, we can now choose one of the arrangements, for instance pairs $A, C$ and $B, D$. Let us consider all possible pairs of parallel lines passing through these points and perpendicular to each other. Each such arrangement is uniquely determined by the angle between the line $ABCD$ and the first pair of straight lines, which varies from $0$ to $\pi$ (at the same time, the slope of the second pair varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$). The ratio of the distance between the straight lines in one pair and the distance in the other pair is a continuous function of the slope, and varies continuously from $0$ through infinity back to $0$. Twice, therefore, the ratio equals $1$, which means that in this case the pairs of parallel lines form a square.
In the second case, we can also obtain two possible squares in an analogous way, so finaly the solution always exists and each time there are four possible squares. $\square$

**Remark**
*An equivalent way to solve this is to draw all the appropriate Thales circles and let a point wander along one of them Then to draw the lines through appropriate points on the given line and check the ratio of the sides of the resulting rectangle.*

**Problem G.2.**

Let $n \in \mathbb{N}$, $n > 1$. Let $X_n$ be the vector space generated by the set of all permutation matrices in the space $M_{n\times n}$ of all $n \times n$ real matrices. Find the dimension of the space $X_n$.

**Remark**

*For a given permutation $\sigma$ of the set $\{1, \ldots, n\}$, the permutation matrix $A_\sigma = [a_{k,m}] \in M_{n\times n}$ is defined by the formula*

$$a_{k,m} = \begin{cases} 1 & \text{if } \sigma(k) = m; \\ 0 & \text{if } \sigma(k) \neq m, \end{cases}$$

*for all $1 \leqslant k, m \leqslant n$.*

**Solution:**

The answer is $\dim(X_n) = n^2 - 2(n-1) = (n-1)^2 + 1$.

Suppose that $n = 2$. We have two permutation matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Consequently $\dim(X_2) = 2$.

Suppose now that $n > 2$. Let $x_1^*, \ldots, x_n^*, y_1^*, \ldots, y_n^* : M_{n\times n} \to \mathbb{R}$ be linear functionals given by the formulas

$$x_j^*([a_{k,m}]) = \sum_{k=1}^{n} a_{j,k} \quad \text{i} \quad y_j^*([a_{k,m}]) = \sum_{k=1}^{n} a_{k,j}$$

for all $1 \leqslant j \leqslant n$ and $[a_{k,m}] \in M_{n\times n}$. We have $x_j^*(A_\sigma) = 1$ and $y_j^*(A_\sigma) = 1$ for all $1 \leqslant j \leqslant n$ and every permutation matrix $A_\sigma$. Hence linear functionals $x_1^* - x_n^*, \ldots, x_{n-1}^* - x_n^*$, $y_1^* - y_n^*, \ldots, y_{n-1}^* - y_n^*$ anihilate space $X_n$. We will show that those functionals are linearly independent in the dual space of $M_{n\times n}$.

Let $E_{i,j} = [a_{k,m}] \in M_{n\times n}$ be given by

$$a_{k,m} = \begin{cases} 1 & \text{if } (k,m) = (i,j) \\ 0 & \text{if } (k,m) \neq (i,j) \end{cases}$$

for all $1 \leqslant i, j \leqslant n$. Suppose that $a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1} \in \mathbb{R}$ are such that

$$\left(\sum_{j=1}^{n-1} a_j(x_j^* - x_n^*)\right) + \left(\sum_{j=1}^{n-1} b_j(y_j^* - y_n^*)\right) = 0.$$

Then

$$\left(\left(\sum_{j=1}^{n-1} a_j(x_j^* - x_n^*)\right) + \left(\sum_{j=1}^{n-1} b_j(y_j^* - y_n^*)\right)\right)\left(\left(\sum_{j=1}^{n-1} E_{k,j}\right) - E_{k,n}\right) = (n-2)a_k = 0$$

for every $1 \leqslant k \leqslant n-1$. Similar consideration shows that $b_k = 0$ for every $1 \leqslant k \leqslant n-1$. Consequently $\dim(X_n) \leqslant n^2 - 2(n-1) = (n-1)^2 + 1$.

We will now show the inequality in the other direction by induction on $n$, which will complete the proof of the fact that $\dim(X_n) = n^2 - 2(n-1) = (n-1)^2 + 1$.

The first step has already been done. Suppose now that $\dim(X_n) \geqslant (n-1)^2 + 1$ for some $n > 1$. Thus we have permutations $\hat{\sigma}_1, \ldots, \hat{\sigma}_{(n-1)^2+1}$ of the set $\{1, \ldots, n\}$ such that matrices $A_{\hat{\sigma}_1}, \ldots, A_{\hat{\sigma}_{(n-1)^2+1}1}$ are linearly independent in the space $M_{n\times n}$. We define a new collection of permutations, this time of the set $\{1, \ldots, n+1\}$. Let $\sigma_j$ be such that $\sigma_j(k) = \hat{\sigma}_j(k)$ and $\sigma_j(n+1) = n+1$ for all $1 \leqslant j \leqslant (n-1)^2 + 1$ and $1 \leqslant k \leqslant n$. Let $\eta_j$ be such that $\eta_j(n+1) = 1$ and $\eta_j(j) = n+1$ for $1 \leqslant j \leqslant n$. Let $\zeta_j$ be such that $\zeta_j(n+1) = j$ for $2 \leqslant j \leqslant n$. The values of the permutations $\eta_j$ and $\zeta_j$ for the other arguments do not matter.

We will show that matrices $A_{\sigma_1}, \ldots, A_{\sigma_{(n-1)^2+1}}, A_{\eta_1}, \ldots, A_{\eta_n}, A_{\zeta_2}, \ldots, A_{\zeta_n}$ are linearly independent in the space $M_{n+1\times n+1}$. Let coefficients $a_1, \ldots, a_{(n-1)^2+1}, b_1, \ldots, b_n, c_2, \ldots, c_n \in \mathbb{R}$ be such that

$$\left(\sum_{j=1}^{(n-1)^2+1} a_j A_{\sigma_j}\right) + \left(\sum_{j=1}^{n} b_j A_{\eta_j}\right) + \left(\sum_{j=2}^{n} c_j A_{\zeta_j}\right) = 0.$$

The $n+1$-th row of the matrix on the left-hand side of the above equality is as follows

$$\left( \sum_{j=1}^{n} b_j, \quad c_2, \quad \dots \quad c_n, \quad \sum_{j=1}^{(n-1)^2+1} a_j \right)$$

So $c_2 = \ldots = c_n = 0$. Consequently, the $n+1$-th column of the same matrix takes the form

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \\ \sum\limits_{j=1}^{(n-1)^2+1} a_j \end{pmatrix}.$$

Hence $b_1 = \ldots = b_n = 0$. The equality $a_1 = \ldots = a_{(n-1)^2+1} = 0$ follows from the induction assumption as

$$\sum_{j=1}^{(n-1)^2+1} a_j A_{\sigma_j} = \begin{pmatrix} \sum\limits_{j=1}^{(n-1)^2+1} a_j A_{\hat{\sigma}_j} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 0 \cdots 0 & \sum\limits_{j=1}^{(n-1)^2+1} a_j \end{pmatrix} = 0,$$

what finishes the proof of the inequality

$$\dim(X_{n+1}) \geqslant \big((n-1)^2 + 1\big) + n + (n-1) = n^2 + 1\,.$$

$\square$

**Remark**

*We can solve this by looking at a linear combination of permutation matrices in the span, changing the coefficient of the identity such that the sum of the coefficients is $1$ and then adding an appropriate multiple of the identity to get the same result. This means that this space is spanned by the identity matrix and the space consisting of "stochastic" matrices without the positivity condition. "Stochastic" matrices can be uniquely defined by freely choosing the entries in the upper left $(n{-}1)\times(n{-}1)$ submatrix and then making sure that sums of all row and column are $1$. Since the identity matrix is clearly linearly independent from this subspace we get the dimension equal to $(n-1)^2+1$.*

**Problem P.1.**

Find all $n \in \mathbb{N}$, $n \geqslant 2$ for which there exist Lebesgue measurable subsets $A_1, \ldots, A_n$ of $[0,1]$ such that

(1) $\lambda(A_k) = \lambda(A_1) > 0$ for every $1 \leqslant k \leqslant n$ and

(2) $\sum_{k=1}^n \lambda(A_k) = \sum_{1 \leqslant j < k \leqslant n} \lambda(A_j \cap A_k)$

where $\lambda$ is the Lebesgue measure on $[0,1]$.

**Solution:**

For $n = 2$ and all measurable sets $A_1, A_2 \subset [0,1]$ we have

$$\lambda(A_1 \cap A_2) \leqslant \lambda(A_1) < \lambda(A_1) + \lambda(A_2).$$

Therefore for $n = 2$ such sets does not exist.

For $n = 3$, we put $A_1 = A_2 = A_3 = A$ where $A$ is any measurable subset of $[0,1]$ such that $\lambda(A) > 0$. Then

$$\sum_{k=1}^{3} \lambda(A_k) = 3\lambda(A) = \sum_{1 \leqslant j < k \leqslant 3} \lambda(A_j \cap A_k).$$

Let now $n \geqslant 4$. We will need the following well known fact

**Lemma**

*For any probability space $(\Omega, \Sigma, P)$ and random variables $f_1, \ldots, f_n$ on this space, we find borel measurable functions $g_1, \ldots, g_n$ on $[0,1]$ with the same joint probability distribution.*

Therefore, we find independent random variables (=borel functions) $h_1, \ldots, h_n$ on the probability space $([0,1], \mathfrak{B}_{[0,1]}, \lambda)$ such that

$$\lambda(h_k = 1) = \frac{2}{n-1}, \quad \text{and} \quad \lambda(h_k = 0) = \frac{n-3}{n-1}$$

for every $1 \leqslant k \leqslant n$. We put

$$A_k = h_k^{-1}(\{1\})$$

for every $1 \leqslant k \leqslant n$. Consequently

$$\sum_{k=1}^{n} \lambda(A_k) = \frac{2n}{n-1} = \binom{n}{2} \frac{2^2}{(n-1)^2} = \sum_{1 \leqslant j < k \leqslant n} \lambda(A_j \cap A_k).$$

$\square$

**Problem P.2.**

In a volleyball match, each ball is served by one team and ends with one team winning one point. The next ball is served by the team that won the previous point. Typically, the probability $p$ of the serving team to win a point is different (smaller) than to lose a point. We assume that the results of each ball are independent. A set ends when one team has at least $25$ points and at least $2$ points more than the other team.

At some moment the score is $24{:}22$ for the team that won the last point. What is the expected number of points of the team that will be victorious in this set?

**Solution:**

Let us denote by $S_i$ the number of balls to the end of the set when serving team has $i$ points more then the other one. (For example the question of the problem is to find $\mathbb{E}S_2$). Let $W$ means winning of serving team ($P(W) = p$). We can formulate following relations:

$$\mathbb{E}S_2 = P(W)\mathbb{E}(S_2|W) + P(W')\mathbb{E}(S_2|W') = p \cdot 1 + (1-p) \cdot (1 + \mathbb{E}S_{-1}), \tag{34.1}$$

$$\mathbb{E}S_1 = P(W)\mathbb{E}(S_1|W) + P(W')\mathbb{E}(S_1|W') = p \cdot 1 + (1-p) \cdot (1 + \mathbb{E}S_0), \tag{34.2}$$

$$\mathbb{E}S_0 = 1 + \mathbb{E}(S_1), \tag{34.3}$$

$$\mathbb{E}S_{-1} = P(W)\mathbb{E}(S_{-1}|W) + P(W')\mathbb{E}(S_{-1}|W') = p \cdot (1 + \mathbb{E}S_0) + (1-p) \cdot 1. \tag{34.4}$$

For example: in (34.3) no matter which team wins a ball, afterwards serving team has one point more; in (34.1) if serving team wins a ball, the game ends, but if not, the other team serves and has one point less. We can see that only two expected values are present in equations (34.2) and (34.3), so assuming $x = \mathbb{E}S_0$ and $y = \mathbb{E}S_1$ we have

$$\begin{cases} y = p + (1-p)(1+x) \\ x = 1 + y \end{cases}$$

so $y = p + (1-p)(2+y) = (2-p) + y - py \iff (2-p) = py \iff y = \frac{2-p}{p} = \frac{2}{p} - 1$. Thus $x = \frac{2}{p}$. Further (see (34.4)) $ES_{-1} = p(1+x) + 1 - p = 3$ and (from (34.1)) $ES_2 = p + (1-p)(1+3) = 4 - 3p$. Formally one should check that expected values are finite. But if we denote $L_n =$ probability of the game longer then $n$, then $L_{n+2} \le (1-p^2)L_n$ for $n > 1$ (if serving team wins 2 balls the game ends) end we know, that

$$\mathbb{E}S_2 = \sum_{n=1}^{\infty} L_n < L_1 + (L_2 + L_3)\sum_{n=1}^{\infty}(1-p^2)^n = L_1 + (L_2 + L_3)\frac{1}{p^2} < \infty.$$

$\square$

**Solution 2.:**

Let $e$ be the expected number of further points from a score of $x : x$ with $x$ at least $24$ ($e$ is $\mathbb{E}S_0$ in the shortlist solution and I agree with this value and its calculation). It can be easily seen (for example by drawing a tree diagram with the scores) that

$$e = 1 + p \cdot 1 + (1-p)(1+e)$$

which gives $pe = 2$ and $e = \frac{2}{p}$. Let $E$ be the expected number of further points from the score $24{:}24$ with “24” serving. Then

$$E = 1 + (1-p)(1 + pe) = 1 + (1-p)(1+2) = 4 - 3p$$

as in the proposed solution. $\square$

**Remark**

*The answer I have found is $26 - p$. The random variable which expectation we are looking for, has mixed Dirac distribution at $25$ and geometric distribution.*