ISTCiM 2024

PROBLEMS

Individual part

(A) Algebra & Combinatorics

Problem A
Let $(R,+,\cdot)$ be a commutative ring. If I and J are two ideals of R then prove that

$$\sqrt{I\cap J}=\sqrt{I}\cap\sqrt{J}$$

and find n such that

$$n\,\mathbb{Z} \;=\; \sqrt{8\,\mathbb{Z}}\cap\sqrt{11\,\mathbb{Z}}\cap\sqrt{2024\,\mathbb{Z}}.$$

Remark
The radical $\sqrt{I}$ of an ideal I is an ideal which consists of all elements in the ring with some power in I, i.e.

$$\sqrt{I}=\{a\in R:\ \underset{n\geqslant 1}{\exists}\ a^n\in I\}.$$

(C) Calculus & Mathematical Analysis

Problem C
Prove that

$$\sum_{n=1}^{\infty}\frac{n^{2024}}{n!}\notin\mathbb{Q}.$$

(E) Equations & Inequalities

Problem E
Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f(A\triangle B)=f(A)\triangle f(B),$$

where $\triangle$ is the symmetric difference of sets: $A\triangle B=(A\setminus B)\cup(B\setminus A)$.
Remark
We define $f(A)=\{x\in\mathbb{R}:\ \underset{a\in A}{\exists}\ f(a)=x\}$ for any subset $A\subset\mathbb{R}$.

(G) Geometry & Linear Algebra

Problem G
Let x_n denote the maximal determinant of an $n\times n$ matrix with entries equal to ±1. Does the sequence $\sqrt[n]{x_n}$ have a finite limit?

(P) Set Theory & Probability

Problem P
Let $(\mathrm{X},\preccurlyeq)$ be a partially ordered set such that for all subsets $A,B\subset\mathrm{X}$ the following property is satisfied

$$\Big(\underset{\substack{x\in A\\ y\in B}}{\forall}\ x\preccurlyeq y\Big)\Longrightarrow\underset{z\in\mathrm{X}}{\exists}\ \Big(\Big(\underset{x\in A}{\forall}\ x\preccurlyeq z\Big)\wedge\Big(\underset{y\in B}{\forall}\ z\preccurlyeq y\Big)\Big).$$

(i) Show that every order preserving function $f\colon\mathrm{X}\to\mathrm{X}$ (i.e. $\underset{x,y\in\mathrm{X}}{\forall}\ x\preccurlyeq y\Rightarrow f(x)\preccurlyeq f(y)$) has a fixed point (i.e. there is an $x_0\in\mathrm{X}$ such that $f(x_0)=x_0$).

(ii) Give an example of X and f, where the property is satisfied only for non-empty subsets A,B of X and f has no fixed point.