Final List of Problems for the ISTCiM 2024

PROBLEMS

Individual part

(A) Algebra & Combinatorics

Problem A
Let $(R, +, \cdot)$ be a commutative ring. If I and J are two ideals of R then prove that

$$\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$$

and find n such that

$$n\,\mathbb{Z} \;=\; \sqrt{8\,\mathbb{Z}} \cap \sqrt{11\,\mathbb{Z}} \cap \sqrt{2024\,\mathbb{Z}}.$$

Remark
The radical $\sqrt{I}$ of an ideal I is an ideal which consists of all elements in the ring with some power in I, i.e.

$$\sqrt{I} = \{a \in R : \underset{n \geqslant 1}{\exists} \; a^n \in I\}\,.$$

(C) Calculus & Mathematical Analysis

Problem C
Prove that

$$\sum_{n=1}^{\infty} \frac{n^{2024}}{n!} \notin \mathbb{Q}.$$

(E) Equations & Inequalities

Problem E
Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(A \triangle B) = f(A) \triangle f(B),$$

where $\triangle$ is the symmetric difference of sets: $A \triangle B = (A \setminus B) \cup (B \setminus A)$.
Remark
We define $f(A) = \{x \in \mathbb{R}\colon \underset{a \in A}{\exists} \; f(a) = x\}$ *for any subset* $A \subset \mathbb{R}$.

(G) Geometry & Linear Algebra

Problem G
Let x_n denote the maximal determinant of an $n \times n$ matrix with entries equal to ± 1. Does the sequence $\sqrt[n]{x_n}$ have a finite limit?

(P) Set Theory & Probability

Problem P
Let $(\mathrm{X}, \preccurlyeq)$ be a partially ordered set such that for all subsets $A, B \subset \mathrm{X}$ the following property is satisfied

$$\left(\underset{\substack{x \in A \\ y \in B}}{\forall} \; x \preccurlyeq y \right) \Longrightarrow \underset{z \in \mathrm{X}}{\exists} \left(\left(\underset{x \in A}{\forall} \; x \preccurlyeq z \right) \wedge \left(\underset{y \in B}{\forall} \; z \preccurlyeq y \right) \right).$$

(i) Show that every order preserving function $f\colon \mathrm{X} \to \mathrm{X}$ (i.e. $\underset{x,y \in \mathrm{X}}{\forall} \; x \preccurlyeq y \Rightarrow f(x) \preccurlyeq f(y)$) has a fixed point (i.e. there is an $x_0 \in \mathrm{X}$ such that $f(x_0) = x_0$).

(ii) Give an example of X and f, where the property is satisfied only for non-empty subsets A, B of X and f has no fixed point.

SOLUTIONS

Individual part

(A) Algebra & Combinatorics

Problem A [*Proposed by*: Pirmyrat Gurbanov and Murat Chashemov *from*: International University for the Humanities and Development]

Let $(R, +, \cdot)$ be a commutative ring. If I and J are two ideals of R then prove that

$$\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$$

and find n such that

$$n\,\mathbb{Z} \;=\; \sqrt{8\,\mathbb{Z}} \cap \sqrt{11\,\mathbb{Z}} \cap \sqrt{2024\,\mathbb{Z}}.$$

Remark

The radical $\sqrt{I}$ of an ideal I is an ideal which consists of all elements in the ring with some power in I, i.e.

$$\sqrt{I} = \{a \in R : \underset{n\geqslant 1}{\exists}\; a^n \in I\}.$$

Solution:

First, we claim that, if $I \subset J$, then $\sqrt{I} \subset \sqrt{J}$. Indeed, if $x \in \sqrt{I}$, then there exists n such that $x^n \in I \subset J$ and $x \in \sqrt{J}$, from which it follows that $\sqrt{I} \subset \sqrt{J}$.

Since $I \cap J \subset I$ and $I \cap J \subset J$, then $\sqrt{I \cap J} \subset \sqrt{I}$ and $\sqrt{I \cap J} \subset \sqrt{J}$, from which it follows that $\sqrt{I \cap J} \subset \sqrt{I} \cap \sqrt{J}$. Now, suppose $x \in \sqrt{I} \cap \sqrt{J}$. Then, there exist two integers m, n such that $x^n \in I$ and $x^m \in J$. On account of the definition of an ideal, we have $x^n \cdot x^m$ belongs to I and to J.So, $x^n \cdot x^m = x^{n+m} \in I \cap J$. Hence, x is an element of $\sqrt{I \cap J}$ and $\sqrt{I} \cap \sqrt{J} \subset \sqrt{I \cap J}$. From the preceding, we get $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$, as desired.

As for the second question, in $\mathbb{Z}$ we have that $\sqrt{2024\,\mathbb{Z}}$ is the set of integers x such that there exists a power x^n which is a multiple of 2024. Since $2024 = 2^3 \cdot 11 \cdot 23$ then a power of x, say x^n, will be divisible by these factors if and only if x is divisible by $2 \cdot 11 \cdot 23 = 506$, and one have that $\sqrt{2024\mathbb{Z}} = 506\mathbb{Z}$. Similarly, $\sqrt{11\,\mathbb{Z}} = 11\,\mathbb{Z}$ and $\sqrt{8\,\mathbb{Z}} = 2\,\mathbb{Z}$.

Finally, we get

$$\sqrt{8\,\mathbb{Z}} \cap \sqrt{11\,\mathbb{Z}} \cap \sqrt{2024\,\mathbb{Z}} = 2\,\mathbb{Z} \cap 11\,\mathbb{Z} \cap 506\,\mathbb{Z} = 506\,\mathbb{Z}$$

as $506\,\mathbb{Z} \subset 2\,\mathbb{Z}$ and $506\,\mathbb{Z} \subset 11\,\mathbb{Z}$ (both 2 and 11 devide 506).

□

(C) Calculus & Mathematical Analysis

Problem C [*Proposed by*: Robert Skiba *from*: Nicolaus Copernicus University in Toruń]

Prove that

$$\sum_{n=1}^{\infty} \frac{n^{2024}}{n!} \notin \mathbb{Q}.$$

Solution:

We put

$$x_k := \frac{1}{e} \sum_{n=1}^{\infty} \frac{n^k}{n!} \text{ for all } k \in \mathbb{N}.$$

It is clear that

$$x_1 = \frac{1}{e} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = \frac{1}{e} \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{e} \cdot e = 1.$$

Now we will show by induction that $x_k \in \mathbb{N}$ for all $k \in \mathbb{N}$. Indeed, let us assume that $x_1, ...x_k \in \mathbb{N}$. We will show that $x_{k+1} \in \mathbb{N}$. We have

$$\begin{aligned}
e \cdot x_{k+1} &= \sum_{n=1}^{\infty} \frac{n^{k+1}}{n!} = \sum_{n=1}^{\infty} \frac{n^k}{(n-1)!} = \sum_{n=0}^{\infty} \frac{(n+1)^k}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!}(n+1)^k \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{k} \binom{k}{m} n^m = \sum_{n=0}^{\infty} \sum_{m=0}^{k} \binom{k}{m} \frac{n^m}{n!} = \sum_{m=0}^{k} \sum_{n=0}^{\infty} \binom{k}{m} \frac{n^m}{n!} \\
&= \sum_{m=0}^{k} \binom{k}{m} \sum_{n=0}^{\infty} \frac{n^m}{n!} = e + \sum_{m=1}^{k} \binom{k}{m} \sum_{n=0}^{\infty} \frac{n^m}{n!} = e + \sum_{m=1}^{k} \binom{k}{m} e x_m \\
&= e\left(1 + \sum_{m=1}^{k} \binom{k}{m} x_m\right),
\end{aligned}$$

which implies that

$$x_{k+1} = 1 + \sum_{m=1}^{k} \binom{k}{m} x_m \in \mathbb{N}.$$

Thus we have proved that $x_k \in \mathbb{N}$ for all k, and hence

$$\sum_{n=1}^{\infty} \frac{n^{2024}}{n!} = e \cdot x_{2024} \notin \mathbb{Q}.$$

This completes the solution. □

Solution 2:

Let $f_k(x) = \underbrace{(\ldots((e^x x)'x)' \ldots x)'}_{k \text{ times}}$. Then $f_k(x) = e^x\, P_k(x)$, where P_k is a polynomial of degree k with integer coefficients.

Proof: by induction on k.

We have $f_1(x) = e^x(x+1)$. If $f_k(x) = e^x\, P_k(x)$, then

$$\begin{aligned} f_{k+1}(x) &= (e^x\, P_k(x)\, x)' = e^x\, P_k(x)\, x + e^x\, P'_k(x)\, x + e^x\, P_k(x) \\ &= e^x \left(P_k(x)\, x + P'_k(x)\, x + P_k(x)\right), \end{aligned}$$

hence $P_{k+1}(x) = P_k(x)\, x + P'_k(x)\, x + P_k(x)$. ◇

We have

$$f_k(x) = \sum_{n=1}^{\infty} \frac{n^k}{n!} x^{k-1}\,.$$

by simply differentiation term by term, as all series are absolutely convergent on the whole $\mathbb{R}$.

As P_{2024} has integer coefficients, $P_{2024}(1) \in \mathbb{Z}$. Hence

$$\sum_{n=1}^{\infty} \frac{n^{2024}}{n!} = f_{2024}(1) = e\, P_{2024}(1) \notin \mathbb{Q}\,.$$

□

(E) Equations & Inequalities

Problem E [*Proposed by*: Leszek Pieniążek *from*: Jagiellonian University]

Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(A \triangle B) = f(A) \triangle f(B),$$

where $\triangle$ is the symmetric difference of sets: $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Remark

We define $f(A) = \{x \in \mathbb{R} \colon \underset{a \in A}{\exists}\ f(a) = x\}$ *for any subset* $A \subset \mathbb{R}$*.*

Solution:

One easily checks, that f must be injective (for sets $A = \{a\}$ and $B = \{b\}$, with $a \neq b$). In an obvious way every injection f fulfils the conditions, as

$$f(A \cup B) = f(A) \cup f(B), \qquad f(A \cap B) = f(A) \cap f(B), \qquad f(A \setminus B) = f(A) \setminus f(B)$$

for any sets A and B. □

(G) Geometry & Linear Algebra

Problem G [*Proposed by*: Robert Skiba *from*: Nicolaus Copernicus University in Toruń]

Let x_n denote the maximal determinant of an $n \times n$ matrix with entries equal to ± 1. Does the sequence $\sqrt[n]{x_n}$ have a finite limit?

Solution:

The answer is negative. We will construct a sequence (A_n) of $2^n \times 2^n$ matrices such that

$$\sqrt[2^n]{\det A_n} \xrightarrow[n \to \infty]{} \infty.$$

Indeed, let $A_0 = (1)$ and

$$A_{n+1} = \begin{pmatrix} A_n & A_n \\ -A_n & A_n \end{pmatrix}$$

for all $n \in \mathbb{N}$. Then

$$\det A_{n+1} = \det \begin{pmatrix} A_n & A_n \\ -A_n & A_n \end{pmatrix} = \det \begin{pmatrix} A_n & A_n \\ 0 & 2A_n \end{pmatrix} = 2^{2^n} (\det A_n)^2.$$

One can show by induction that

$$\det A_n = 2^{n2^{n-1}} \text{ for all } n \geq 0.$$

Finally, one has

$$\sqrt[2^n]{\det A_n} = \sqrt[2^n]{2^{n2^{n-1}}} = 2^{n/2} \xrightarrow[n\to\infty]{} \infty.$$

This completes the solution. □

(P) Set Theory & Probability

Problem P [*Proposed by*: Marcin J. Zygmunt *from*: University of Silesia, Katowice (Poland)]
Let $(\mathrm{X}, \preccurlyeq)$ be a partially ordered set such that for all subsets $A, B \subset \mathrm{X}$ the following property is satisfied

$$\left(\underset{\substack{x\in A\\ y\in B}}{\forall} x \preccurlyeq y \right) \Longrightarrow \underset{z\in \mathrm{X}}{\exists} \left(\left(\underset{x\in A}{\forall} x \preccurlyeq z \right) \wedge \left(\underset{y\in B}{\forall} z \preccurlyeq y \right) \right).$$

(i) Show that every order preserving function $f\colon \mathrm{X} \to \mathrm{X}$ (i.e. $\underset{x,y\in \mathrm{X}}{\forall} x \preccurlyeq y \Rightarrow f(x) \preccurlyeq f(y)$) has a fixed point (i.e. there is an $x_0 \in \mathrm{X}$ such that $f(x_0) = x_0$).

(ii) Give an example of X and f, where the property is satisfied only for non-empty subsets A, B of X and f has no fixed point.

Solution:
Ad (i) Let $C = \left\{ x\in \mathrm{X}\colon\ x \preccurlyeq f(x) \right\}$ and let $D = \left\{ x\in \mathrm{X}\colon \underset{y\in C}{\forall} y \preccurlyeq x \right\}$. By the property there exists $z_o \in \mathrm{X}$ such that $\underset{x\in C}{\forall} x \preccurlyeq z_0$ and $\underset{y\in D}{\forall} z_0 \preccurlyeq y$. Note that if the set C were empty, then D would be equal to X, so by the property z_0 would have to be the smallest element of the set X. In that case, $z_0 \preccurlyeq f(z_0) \in X$, which means that $z_0 \in C$. Hence $C \neq \varnothing$.
We will show that $f(z_0) = z_0$, or more precisely $f(z_0) \preccurlyeq z_0$ and $f(z_0) \succcurlyeq z_0$.
We have $\underset{x\in C}{\forall} x \preccurlyeq z_0 \Rightarrow \underset{x\in C}{\forall} x \preccurlyeq f(x) \preccurlyeq f(z_0)$, hence $f(z_0) \in D$. Moreover, as $\underset{y\in D}{\forall} z_0 \preccurlyeq y$ then $z_0 \preccurlyeq f(z_0)$, hence $z_0 \in C$.
On the other hand applying function f to $x \preccurlyeq f(x)$ gives $f(x) \preccurlyeq f(f(x))$ for such x's, so $f(C) \subset C$ and $f(z_0) \in C$. Thus $f(z_0) \preccurlyeq z_0$.
So by the antisymmetry property of ordering $\preccurlyeq$ we get $f(z_0) = z_0$.

Ad (ii) One can observe that the condition for non-empty sets will always be satisfied whenever each subset of X, which is bounded above (respectively, bounded below), has the largest (respectively, smallest) element. An example of such a set are the integers with the usual order $(\mathbb{Z}, \leq)$. An order-preserving function $f\colon \mathbb{Z} \to \mathbb{Z}$ that does not have a fixed point is, for example, the shift $f(n) = n+1$.

Remark
In the analogical way it can be proved the existence of sup *and* inf *for all subsets in* X. *Then the thesis follows from the well known version of the Banach's Fixpoint Lemma.*

□