Individual Part

Algebra & Combinatorics

Problem A
Show that 2021^n can be expressed as difference of two squares of natural numbers for every positive integer n (i.e. $n = 1, 2, \ldots$).

Calculus & Mathematical Analysis (with topology and set theory)

Problem C
Find the limit $\lim\limits_{n\to\infty} (2\sqrt[n]{2} - 1)^n$.

Equations & Inequalities (including differential equations)

Problem E
Solve for $x \in \mathbb{R}$ the equation $\displaystyle\int_0^{+\infty} \frac{dt}{\left(1 + t^{\,x}\right)^x} = 1\,.$

Geometry & Linear Algebra

Problem G
We have the following L-system:

$$\begin{aligned} \omega &: C \\ p_A &: A \to A \\ p_B &: B \to B^2A \\ p_C &: C \to C^3BA. \end{aligned}$$

This means that in the zeroth step, we have the string C. If a letter X occurs n times in a row, we may write X^n (we assume of course, that the length of X^n still equals to n). Having obtained a string in step n to obtain the corresponding string in step $n+1$, we have to replace each letter with the string on the right-hand side in the corresponding rule.
More precisely assume that $s_1 \cdots s_m$ is the string obtained in step n, where $s_i \in \{A, B, C\}$ for $i \in \{1, \ldots, m\}$. For a character $s \in \{A, B, C\}$ let us denote

$$\overline{s} = \begin{cases} A, & \text{if } s = A \\ B^2A, & \text{if } s = B \\ C^3BA, & \text{if } s = C. \end{cases}$$

Then the string in step $n+1$ is $\overline{s_1} \cdots \overline{s_m}$. This means in the first step, we have C^3BA and in the next step $C^3BAC^3BAC^3BAB^2A^2$, and so on.
Find the formula for the length of the obtained string in step n.

Measure Theory & Probability

Problem P
Assume that we are given two football players P_1 and P_2. Let us denote by C_i the average of scored goals per match for P_i while playing for a club and N_i the average of scored goals per match scored for the national team. Finally for both players P_i, we denote by T_i the average of scored goals per match regardless whether the match was for the club or the national team. We tacitly assume here that both players played for a club and also the national team. Is it possible that $C_1 < C_2$ and $N_1 < N_2$, but $T_2 < T_1$?

Team Part

Algebra & Combinatorics

Problem A.1
Let $f,\ g$ be two bijections of $\mathbb{N}$ such that $\forall_{n\in\mathbb{N}}\ f^{-1}\big(g(n)\big) + g^{-1}\big(f(n)\big) = 2n$. Prove that $f = g$.
Remark
The solution does not depend on the assumption that $0 \in \mathbb{N}$

Problem A.2
Let R be a ring with unity, non necessary commutative, of characteristic 0, and let $a, b \in R \setminus \{1\}$, such that $a^2 = b$ and $b^2 = a$.
Determine conditions on a and b for which there exists an $x \in R$ such that $ax + xb = 2$, and express the formula for x in terms of a and b.

Calculus & Mathematical Analysis (with topology and set theory)

Problem C.1
Suppose that $f\colon [0,1] \to \mathbb{R}$ is continuous, differentiable on $(0,1)$, and satisfies $f(0) = 1$ as well as $f(1) = 0$. Show that there exists some $x \in (0,1)$ such that

$$f'(x) = \frac{2}{e^{-1} - 1} x e^{-x^2}.$$

Problem C.2
Let $x_0 = 2021$ and $x_n = \dfrac{1}{x_0 + \cdots + x_{n-1}}$ for $n = 1, 2, \ldots$. Study convergence of series $\sum\limits_{n=0}^{\infty} x_n$ and $\sum\limits_{n=0}^{\infty} x_n^2$.

Equations & Inequalities (including differential equations)

Problem E.1
Let $f : \mathbb{R} \to (0, +\infty)$ be a continuous periodic function with period T (i.e. $f(x+T) = f(x)$ for all $x \in \mathbb{R}$). Prove that

$$\int_0^T \frac{f(x)}{f(nx)}\, dx \;\geq\; T \text{ for all } n \in \mathbb{Z}_+ .$$

Problem E.2
Prove that the following inequality holds

$$\sum_{n,m=1}^{N} (-1)^{n+m} \, \frac{n^n \, m^m}{(n+m)^2} > 0$$

for any positive integer N.

Geometry & Linear Algebra

Problem G.1
Let a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ have two different real eigenvalues. Express the value of the cosine of the angle between corresponding eigenvectors in terms of a, b, c, d.

Problem G.2
Assume that a symmetric matrix $A \in M_{n\times n}(\mathbb{R})$ satisfies the following equation:

$$A^5 - A^4 + 2A^3 - 2A^2 + A = I,$$

where I denotes the identity matrix. Show that $A = I$.

Measure Theory & Probability

Problem P.1
Two candidates, Aiden and Bump, are running for president. The vote, in which $2n$ valid votes were cast (a valid vote is a vote cast for one of the candidates), ended in a tie. What is the probability that the leading person changed only once during the vote count (a tie does not count as a change of lead).

Problem P.2
Numbers $0, 1, 2, \ldots, 2021$ are distributed randomly on a circle. We start from 2021 and go along the circle. Every encountered number which is smaller then all other numbers is removed. (So we first remove 0, then 1, and so on up to 2021.) We stop after removing all numbers. What is the expected value of rotation along the circle to be made?

SOLUTIONS

Problem A
Show that 2021^n can be expressed as difference of two squares of natural numbers for every positive integer n (i.e. $n = 1, 2, \ldots$).

Remark
It's rather an easy problem, but still it needs some tricky approach.
For the individual part.

Solution:
It can be easily verified that $45^2 = 9^2 \cdot 5^2 = 81 \cdot 25 = 2025$, hence

$$2021 = 45^2 - 4 = 45^2 - 2^2 = 43 \cdot 47\,.$$

Now, as we have

$$(a \pm b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} (\pm 1)^k b^k = \sum_{k=0}^{[n/2]} \binom{n}{2k} a^{n-2k} b^{2k} \pm \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} a^{n-2k-1} b^{2k+1}$$

we get

$$2021^n = (45-2)^n\,(45+2)^n = \left(\sum_{k=0}^{[n/2]} \binom{n}{2k} 45^{n-2k} 2^{2k} \right)^2 - \left(\sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} 45^{n-2k-1} 2^{2k+1} \right)^2,$$

and both components in the last difference are greater than zero. $\square$

Problem A.1
Let f, g be two bijections of $\mathbb{N}$ such that $\forall_{n\in\mathbb{N}}\ f^{-1}\big(g(n)\big) + g^{-1}\big(f(n)\big) = 2n$. Prove that $f = g$.

Remark
The solution does not depend on the assumption that $0 \in \mathbb{N}$

Solution:
Let $\phi = f^{-1} \circ g$. Of course ϕ is a bijection of $\mathbb{N}$ and $\phi^{-1} = g^{-1} \circ f$. Moreover it satisfies $\phi(n) + \phi^{-1}(n) = 2n$ for all $n \in \mathbb{N}$. Hence $\phi(n) - n = n - \phi^{-1}(n)$. Substituting $\phi^k(n)$ for n gives $\phi^{k+1}(n) - \phi^k(n) = \phi^k(n) - \phi^{n-1}(n)$ for any $k \in \mathbb{Z}$, so $\phi^{k+1}(n) - \phi^k(n) = \phi(n) - n$. Hence $\phi^k(n) = n + k\big(\phi(n) - n\big)$ for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$.
But $\phi^k(n) \geqslant 0$ for any $k \in \mathbb{Z}$, so $\phi(n) - n = 0$, which means that $f(n) = g(n)$ for all $n \in \mathbb{N}$. $\square$

Problem A.2

Let R be a ring with unity, non necessary commutative, of characteristic 0, and let $a, b \in R \setminus \{1\}$, such that $a^2 = b$ and $b^2 = a$.
Determine conditions on a and b for which there exists an $x \in R$ such that $ax + xb = 2$, and express the formula for x in terms of a and b.

Remark

It follows from the principle of separability that in any ring with unity, addition must be commutative. Namely on one hand we have

$$(x+1)(y+1) = x(y+1) + x(y+1) = xy + x + y + 1\,,$$

and on the other hand

$$(x+1)(y+1) = (x+1)y + (x+1) = xy + y + x + 1\,.$$

Hence $x + y = y + x$ for every $x, y \in R$.

Solution:

Assume that such x exists. Then multiplying the equation

$$ax + xb = 2 \tag{18.1}$$

on the right by b and on the left by a gives

$$axb + xb^2 = 2b \qquad \text{and} \qquad a^2x + axb = 2a$$

hence

$$xa - 2b = xb^2 - 2b = -axb = a^2x - 2a = bx - 2a$$

or

$$xa - bx = 2b - 2a\,. \tag{18.2}$$

Once again multiplying (18.2) this time on the right by a and on the left by b gives

$$xa^2 - bxa = 2ba - 2a^2 \qquad \text{and} \qquad bxa - b^2x = 2b^2 - 2ba$$

hence

$$xb + 2b = xa^2 + 2a^2 = 2ba + bxa = b^2x + 2b^2 = ax + 2a$$

or

$$ax - xb = 2b - 2a\,. \tag{18.3}$$

Adding (18.1) and (18.3) shows

$$2ax = 2 + 2b - 2a = 2(1 + b - a)\,,$$

so

$$ax = 1 + b - a\,. \tag{18.4}$$

Now (as we have $a^4 = (a^2)^2 = b^2 = a$) multiplying (18.4) on the left by a^3 leads to

$$ax = a^4x = a^3 + a^3b - a^4 = a^3 + aa^2b - a = a^3 + b - a\,.$$

Putting it together with (18.4) shows that $1 = a^3$ (so a^{-1} exists in R, i.e. $a^{-1} = b$, and it is the necessary condition for the existence of x). Hence

$$x = a^{-1}ax = a^{-1}(1 + b - a) = b + a - 1\,.$$

To complete solution it is enough to verify that such x satisfies (18.1). $\square$

Problem C
Find the limit
$$\lim_{n\to\infty}(2\sqrt[n]{2}-1)^n.$$

Solution:
First, observe that
$$0<(\sqrt[n]{2}-1)^2=\sqrt[n]{4}-2\sqrt[n]{2}+1\Longrightarrow 2\sqrt[n]{2}-1<\sqrt[n]{4}.$$
Thus
$$(2\sqrt[n]{2}-1)^n<(\sqrt[n]{4})^n=4.$$
On the other hand, one has
$$(2\sqrt[n]{2}-1)^n=4\left(1-\left(1-\frac{1}{\sqrt[n]{2}}\right)^2\right)^n.$$
Next, Bernoulli's inequality implies that
$$\left(1-\left(1-\frac{1}{\sqrt[n]{2}}\right)^2\right)^n\geqslant 1-n\left(1-\frac{1}{\sqrt[n]{2}}\right)^2=1-n\frac{(\sqrt[n]{2}-1)^2}{\sqrt[n]{4}} \tag{19.1}$$
and
$$2=(\sqrt[n]{2}-1+1)^n\geqslant 1+n(\sqrt[n]{2}-1)>n(\sqrt[n]{2}-1)\Longrightarrow(\sqrt[n]{2}-1)^2<\frac{4}{n^2}. \tag{19.2}$$
Consequently, taking into account (19.1) and (19.2) gives
$$(2\sqrt[n]{2}-1)^n>4\left(1-\frac{4}{n\sqrt[n]{4}}\right).$$
Putting all the required inequalities together shows that
$$4\left(1-\frac{4}{n\sqrt[n]{4}}\right)<(2\sqrt[n]{2}-1)^n<4.$$
Finally, by using the squeeze theorem to the above inequalities, we conclude that
$$\lim_{n\to\infty}(2\sqrt[n]{2}-1)^n=4.$$

□

Solution no.2:
Let $a_n=2(\sqrt[n]{2}-1)$. Then $1+a_n=2\sqrt[n]{2}-1$, $\lim\limits_{n\to\infty}a_n=0$ and $\lim\limits_{n\to\infty}na_n=2\ln 2$. Hence
$$\begin{aligned}(2\sqrt[n]{2}-1)^n&=\exp\ln\left(2\sqrt[n]{2}-1\right)^n=\exp\left(n\,a_n\,\ln\left(1+a_n\right)^{1/a_n}\right)\\&\xrightarrow{n\to\infty}\exp(2\,\ln 2\,\ln e)=4\,.\end{aligned}$$

□

Problem C.1
Suppose that $f\colon [0,1] \to \mathbb{R}$ is continuous, differentiable on $(0,1)$, and satisfies $f(0) = 1$ as well as $f(1) = 0$. Show that there exists some $x \in (0,1)$ such that

$$f'(x) = \frac{2}{e^{-1} - 1} x e^{-x^2}.$$

Solution:
Note that $-2xe^{-x^2}$ is the derivative of e^{-x^2}. We define $g\colon [0,1] \to \mathbb{R}$ by $g(x) = e^{-x^2}$. In this way, we see that have to show that

$$g'(x) = (1 - e^{-1}) f'(x).$$

Remembering Cauchy's mean value theorem, which we may apply for $0 \leq a \leq b \leq 1$ as f and g are continuous on $[a,b]$ and differentiable on (a,b), there is $x \in (a,b)$ such that

$$(f(b) - f(a)) g'(x) = (g(b) - g(a)) f'(x).$$

We try with $b = 1$ and $a = 0$:

$$-g'(x) = (e^{-1} - 1) f'(x).$$

Thus

$$f'(x) = \frac{g'(x)}{1 - e^{-1}} = \frac{2}{e^{-1} - 1} x e^{-x^2}.$$

$\square$

Problem C.2
Let $x_0 = 2021$ and $x_n = \dfrac{1}{x_0 + \cdots + x_{n-1}}$ for $n = 1, 2, \ldots$.
Verify convergence of series $\sum\limits_{n=0}^{\infty} x_n$ and $\sum\limits_{n=0}^{\infty} x_n^2$.

Solution:
Both series diverge to the infinity and their convergence does not depend on the value of the first term $x_0 > 0$.
It is obvious that $x_n > 0$ and $x_n > x_{n+1}$ for all $n \in \mathbb{N}$. For the second inequality we can see also that

$$x_n - x_{n+1} = \frac{x_n}{\big(\sum\limits_{k=0}^{n-1} x_k\big)\big(\sum\limits_{k=0}^{n} x_k\big)} = x_n^2 x_{n+1} > 0\,. \tag{21.1}$$

So the sequence (x_n) is convergent to the nonnegative limit.
On the other hand $x_{n+1}\Big(\sum\limits_{k=0}^{n} x_k\Big) = 1$, so in the limit we get

$$\sum_{n=0}^{\infty} x_n = \frac{1}{\lim\limits_{n\to\infty} x_n}\,.$$

If the series $\sum\limits_{n=0}^{\infty} x_n$ had been convergent, the limit of x_n would be 0, which is not possible by above equality. Hence the series $\sum\limits_{n=0}^{\infty} x_n$ diverges to the infinity (and $\lim\limits_{n\to\infty} x_n = 0$).
As for the second series in question, the equation (21.1) gives $\dfrac{x_n}{x_{n+1}} = 1 + x_n^2$, hence

$$\prod_{n=0}^{\infty}\big(1 + x_n^2\big) = \lim_{n\to\infty} \frac{x_0}{x_{n+1}} = \infty\,.$$

Thus the series $\sum\limits_{n=0}^{\infty} x_n^2$ diverges by the following lemma.

Lemma 1
Let $a_n > 0$ for all $n \in \mathbb{N}$. Then the infinite product $\prod\limits_{n=0}^{\infty}\big(1 + a_n\big)$ converges iff the infinite sum $\sum\limits_{n=0}^{\infty} a_n$ converges.

Proof: On the one hand we have

$$1 + \sum_{n=0}^{\infty} a_n \leqslant \prod_{n=0}^{\infty}\big(1 + a_n\big)\,.$$

On the other hand the inequality $1 + x \leqslant e^x$ gives

$$\prod_{n=0}^{\infty}\big(1 + a_n\big) \leqslant \exp\Big(\sum_{n=0}^{\infty} a_n\Big)\,.$$

The thesis follow by the monotone convergence theorem.

◇

□

Problem E

Solve for $x \in \mathbb{R}$ the equation $\displaystyle\int\limits_0^{+\infty} \frac{dt}{\left(1+t^{\,x}\right)^x} = 1\,.$

Solution:
The equation has meaning only for $x > 1$. Indeed, we have

$$\lim_{t\to+\infty} \frac{1}{\left(1+t^{\,x}\right)^x} = 1$$

for $x \leqslant 0$, so the integral on the left-hand side diverges. As for the case $x > 0$ we have

$$\frac{1}{\left(1+t^{\,x}\right)^x} \sim \frac{1}{t^{\,x^2}}$$

for sufficiently large t. Thus the integral converges for $x^2 > 1$, hence $x > 1$.

The change of variable $u = \dfrac{1}{1+t^x}$ transforms the left-hand side to

$$\frac{1}{x}\int\limits_0^1 (1-u)^{1/x-1}\, u^{x-1-1/x}\, du\,. \tag{22.1}$$

Note that

$$\frac{1}{x}\int\limits_0^1 (1-u)^{1/x-1}\, du \;=\; -(1-u)^{1/x}\Bigg|_{u=0}^{u=1} \;=\; 1$$

for any $x > 0$. Hence if $x-1-\frac{1}{x} > 0$ then $0 < u^{x-1-1/x} < 1$ for $u \in (0,1)$, so the integral (22.1) is strictly less than 1 as well. And if $x-1-\frac{1}{x} < 0$ then $u^{x-1-1/x} > 1$ for $u \in (0,1)$, so in this case the integral (22.1) is strictly bigger than 1. Thus the unique possibility is $x-1-\frac{1}{x} = 0$, i.e. $x^2 = 1+x$. It is well-known equation for the golden ratio $\varphi = \dfrac{1+\sqrt{5}}{2}$. The other solution is smaller than 1.

It remains only to check that indeed the equation holds for $x = \varphi = \dfrac{1+\sqrt{5}}{2}$. Indeed, the same change of variable leads to

$$\begin{aligned}\int\limits_0^{+\infty} \frac{dt}{\left(1+t^{\,\varphi}\right)^\varphi} &= \frac{1}{\varphi}\int\limits_0^1 (1-u)^{1/\varphi-1}\, u^{\varphi-1-1/\varphi}\, du \\ &= \frac{1}{\varphi}\int\limits_0^1 (1-u)^{1/\varphi-1}\, du \;=\; 1\,.\end{aligned}$$

$\square$

Problem E.1
Let $f:\mathbb{R}\to(0,+\infty)$ be a continuous periodic function with period T (i.e. $f(x+T)=f(x)$ for all $x\in\mathbb{R}$). Prove that

$$\int_0^T \frac{f(x)}{f(nx)}\,dx \;\geq\; T \text{ for all } n\in\mathbb{Z}_+\,.$$

Solution:
We have

$$\int_0^T \frac{f(x)}{f(nx)}\,dx \;=\; \frac{1}{m}\int_0^{mT} \frac{f(x)}{f(nx)}\,dx \;=\; \int_0^T \frac{f(mx)}{f(nmx)}\,dx\,.$$

So

$$\int_0^T \frac{f(x)}{f(nx)}\,dx \;=\; \frac{1}{N}\int_0^T \Big(\frac{f(x)}{f(nx)}+\frac{f(nx)}{f(n^2x)}+\cdots+\frac{f(n^{N-1}x)}{f(n^Nx)}\Big)\,dx\,.$$

The logarithm function is concave, so by Jensen's inequality[1] we have

$$\log\frac{1}{T}\int_0^T \frac{\frac{f(x)}{f(nx)}+\cdots+\frac{f(n^{N-1}x)}{f(n^Nx)}}{N}\,dx \geqslant \frac{1}{T}\int_0^T \log\frac{\frac{f(x)}{f(nx)}+\cdots+\frac{f(n^{N-1}x)}{f(n^Nx)}}{N}\,dx$$

$$\geqslant \frac{1}{T}\int_0^T \frac{\log\frac{f(x)}{f(nx)}+\cdots+\log\frac{f(n^{N-1}x)}{f(n^Nx)}}{N}\,dx = \frac{1}{T}\int_0^T \frac{\log f(x)-\log f(n^Nx)}{N}\,dx\,.$$

The function f is continuous on closed interval $[0,T]$, hence it bounded, i.e. there exist $C,c>0$ such that $0<c\leqslant f(x)\leqslant C<+\infty$ for all $x\in[0,T]$. The last integral is then bounded by

$$\left|\frac{1}{T}\int_0^T \frac{\log f(x)-\log f(n^Nx)}{N}\,dx\right| \leqslant \frac{C-c}{N} \xrightarrow{N\to\infty} 0\,.$$

Finally

$$\int_0^T \frac{f(x)}{f(nx)}\,dx \;=\; T\,\exp\left(\log\frac{1}{T}\int_0^T \frac{f(x)}{f(nx)}dx\right)$$

$$\geqslant T\exp\left(\frac{1}{T}\int_0^T \frac{\log f(x)-\log f(n^Nx)}{N}\,dx\right) \longrightarrow T\,e^0 \;=\; T\,.$$

□

Solution no.2:
We have

$$\int_0^T \frac{f(x)}{f(nx)}\,dx \;=\; \frac{1}{m}\int_0^{mT} \frac{f(x)}{f(nx)}\,dx \;=\; \int_0^T \frac{f(mx)}{f(nmx)}\,dx\,.$$

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[1]Jensen's integral inequality for a concave function φ states that $\varphi\left(\frac{1}{b-a}\int_a^b F(x)\,dx\right)\geqslant\frac{1}{b-a}\int_a^b\varphi\big(F(x)\big)\,dx$ for a non-negative Lebesgue-integrable function F.

So

$$\int_0^T \frac{f(x)}{f(nx)}\,dx = \frac{1}{N}\int_0^T \Big(\frac{f(x)}{f(nx)} + \frac{f(nx)}{f(n^2x)} + \cdots + \frac{f(n^{N-1}x)}{f(n^Nx)}\Big)\,dx$$

$$\geq \int_0^T \Big(\frac{f(x)}{f(nx)}\cdots\frac{f(n^{N-1}x)}{f(n^Nx)}\Big)^{1/N} dx = \int_0^T \Big(\frac{f(x)}{f(n^Nx)}\Big)^{1/N} dx\,,$$

where the inequality holds by the inequality between arithmetic and geometric means. Let us now choose any $\varepsilon > 0$, and let $A_\varepsilon = \big\{x \in [0,T] \mid 0 < c \leqslant f(x) \leqslant C < +\infty\big\}$. We can choose c and C, such that the Lebesgue measure $\big|[0,T] \setminus A_\varepsilon\big| < \frac{1}{2}\varepsilon$. Let moreover $B_{\varepsilon,N} = \big\{x \in [0,T] \mid 0 < c \leqslant f(n^Nx) \leqslant C < +\infty\big\}$. Due to the periodicity of the function f we have as well $\big|[0,T] \setminus B_{\varepsilon,N}\big| < \frac{1}{2}\varepsilon$. Finally let $E = A_\varepsilon \cap B_{\varepsilon,N}$. We have $\big|E\big| > T - \varepsilon$. Now let N be so large that $\Big(\dfrac{c}{C}\Big)^{1/N} > 1 - \varepsilon$. Then

$$\int_0^T \Big(\frac{f(x)}{f(n^Nx)}\Big)^{1/N} dx \geq \int_E \Big(\frac{c}{C}\Big)^{1/N} dx > (1-\varepsilon)(T-\varepsilon)\,.$$

Since ε was arbitrary, the inequality is proved. □

Remark
The second solution does not require the assumption of continuity of the function f, it's enough only to assume the integrability of f and the inverse function f^{-1} on the interval $[0,T]$.

Remark
One may also ask when the equality is reached.
The equality in the above inequality is obtained if and only if $f(x) = f(nx)$ almost everywhere.
Indeed, assuming that $\displaystyle\int_0^T \frac{f(x)}{f(nx)}\,dx = T$ and applying the result to the function $f(x)^{1/2}$ gives

$$T \leqslant \int_0^T \Big(\frac{f(x)}{f(nx)}\Big)^{1/2} dx \leqslant \Big(\int_0^T \frac{f(x)}{f(nx)}\,dx\Big)^{1/2}\Big(\int_0^T 1^2\,dx\Big)^{1/2} = T\,.$$

Hence the second inequality must be equality, so this implies the linear dependence of $\Big(\frac{f(x)}{f(nx)}\Big)^{1/2}$ and 1. Thus $f(x) = f(nx)$ almost everywhere.

Problem E.2
Prove that the following inequality holds

$$\sum_{n,m=1}^{N} (-1)^{n+m} \frac{n^n \, m^m}{(n+m)^2} > 0$$

for any positive integer N.
Remark
The above inequality for this sum is related to the fact that the sequence $\frac{1}{n^2}$ is positive definite. The Author hoped to see another, more elementary solution.

Remark
The more general inequality holds, i.e.

$$\sum_{n,m=1}^{N} z^{n-m} \frac{a_n \, a_m}{(n+m)^2} > 0$$

for every integer $N > 0$, every complex number $|z| = 1$ and every sequence (a_n).

Solution:
It is known that

$$\int_0^1 x^{n-1} \log x \, dx = -\frac{1}{n^2} .$$

Define now a function $f_N(x) = \sum_{n=1}^{N} (-1)^n n^n x^{n-1/2}$. The function f_N is well-defined and continuous on the interval $(0,1)$. Hence

$$\sum_{n,m=1}^{N} (-1)^{n+m} \frac{n^n \, m^m}{(n+m)^2} = \sum_{n,m=1}^{N} (-1)^{n+m} \, n^n \, m^m \int_0^1 (-x^{n+m-1} \log x) \, dx$$

$$= \int_0^1 \Big(\sum_{n,m=1}^{N} (-1)^n \, n^n \, x^{n-1/2} \cdot (-1)^m \, m^m \, x^{m-1/2} \Big) (-\log x) \, dx = \int_0^1 f_N(x)^2 (-\log x) \, dx > 0 .$$

The last inequality holds because the function under the integral, i.e. $f_N(x)^2(-\log x)$ is nonnegative on $(0,1)$ and it does not equal to 0 everywhere. $\square$

Problem G

We have the following L-system:

$$\begin{aligned}
\omega &: C \\
p_A &: A \to A \\
p_B &: B \to B^2A \\
p_C &: C \to C^3BA.
\end{aligned}$$

This means that in the zeroth step, we have the string C. If a letter X occurs n times in a row, we may write X^n (we assume of course, that the length of X^n still equals to n). Having obtained a string in step n to obtain the corresponding string in step $n+1$, we have to replace each letter with the string on the right-hand side in the corresponding rule.

More precisely assume that $s_1 \cdots s_m$ is the string obtained in step n, where $s_i \in \{A, B, C\}$ for $i \in \{1, \ldots, m\}$. For a character $s \in \{A, B, C\}$ let us denote

$$\overline{s} = \begin{cases} A, & \text{if } s = A \\ B^2A, & \text{if } s = B \\ C^3BA, & \text{if } s = C. \end{cases}$$

Then the string in step $n+1$ is $\overline{s_1} \cdots \overline{s_m}$. This means in the first step, we have C^3BA and in the next step $C^3BAC^3BAC^3BAB^2A^2$, and so on.

Find the formula for the length of the obtained string in step n.

Solution:

Let us write a_n, b_n, and c_n for the amount of occurrences of the letter A, B, and C in step n, respectively. Defining v_n as

$$v_n = \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix},$$

we thus have

$$v_{n+1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} v_n.$$

Let us set

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

Then, we have

$$v_n = A^n v_0,$$

where

$$v_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Our next goal is to determine the Jordan canonical form J of A. We see that, choosing $\lambda \in \mathbb{R}$ and denoting by I the identity,

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda)(3-\lambda).$$

Hence the eigenvalues are 1, 2, and 3. We compute corresponding eigenvectors, starting with $\lambda = 1$:

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We see that we can choose

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Now for $\lambda = 2$:

$$\begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

A possible choice is

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Now, for $\lambda = 3$, we have that

$$\begin{pmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This time, we choose

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Writing $A = SJS^{-1}$, we have $S^{-1}AS = J$ and see that

$$S = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have $\det(S) = 1$ and

$$S^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

From here, we obtain that

$$\begin{aligned} A^n &= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2^n & -2^n \\ 0 & 0 & 3^n \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2^n - 1 & 3^n - 2^n \\ 0 & 2^n & 3^n - 2^n \\ 0 & 0 & 3^n \end{pmatrix}. \end{aligned}$$

Therefore,

$$v_n = \begin{pmatrix} 1 & 2^n - 1 & 3^n - 2^n \\ 0 & 2^n & 3^n - 2^n \\ 0 & 0 & 3^n \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3^n - 2^n \\ 3^n - 2^n \\ 3^n \end{pmatrix}.$$

The length l_n in step n is the sum of the components of the vector: $l_n = 3^{n+1} - 2^{n+1}$. $\square$

Problem G.1
Let a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ have two different real eigenvalues. Express the value of the cosine of the angle between corresponding eigenvectors in terms of a, b, c, d.

Solution:
Let us denote $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The orthogonal change of variables does not change the angle between eigenvalues. So we can choose another basis in such a way that $OAO^T = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$ where $O = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ is an orthogonal matrix, i.e. the equality $x^2 + y^2 = 1$ is satisfied. Hence

$$OTO^T = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

$$= \begin{pmatrix} ax^2 + dy^2 + (b+c)yx & (d-a)xy + cx^2 - by^2 \\ (d-a)xy + bx^2 - cy^2 & ay^2 + dx^2 - (b+c)xy \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}.$$

As $(d-a)xy + bx^2 - cy^2 = 0$ we have $\beta = (d-a)xy + cx^2 - by^2 = (b-c)(x^2+y^2)$, so $\beta = b - c$. The numbers α and γ are two eigenvalues, the first corresponding to the eigenvector $(1, 0)$, and the second – to the vector (x, y). The equality $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathsf{x} \\ \mathsf{y} \end{pmatrix} = \gamma \begin{pmatrix} \mathsf{x} \\ \mathsf{y} \end{pmatrix}$ gives $\beta \mathsf{y} = (\gamma - \alpha)\mathsf{x}$, hence the value of cosinus between the eigenvectors $(1, 0)$ and (x, y) equals to $\frac{\beta}{\sqrt{\beta^2 + (\gamma-\alpha)^2}}$.
But $(\gamma - \alpha)^2 = (\gamma + \alpha)^2 - 4\gamma\alpha = (\mathrm{tr}A)^2 - \det A = (a+d)^2 - 4(ad - bc) = (a-d)^2 + 4bc$, hence $\beta^2 + (\gamma - \alpha)^2 = (b-c)^2 + (a-d)^2 + 4bc = (a-d)^2 + (b+c)^2$. Finaly the value of cosinus between the eigenvectors equals to $\dfrac{b-c}{\sqrt{(a-d)^2 + (b+c)^2}}$. □

Problem G.2
Assume that a symmetric matrix $A \in M_{n\times n}(\mathbb{R})$ satisfies the following equation:

$$A^5 - A^4 + 2A^3 - 2A^2 + A = I,$$

where I denotes the identity matrix. Show that $A = I$.

Solution:
Let

$$w(x) = x^5 - x^4 + 2x^3 - 2x^2 + x - 1.$$

It is not hard to see that $w(1) = 0$ and hence

$$w(x) = (x^4 + 2x^2 + 1)(x - 1).$$

Consequently, $w(x)$ has only one real root. Since A is symmetric, it follows that $\sigma(A) \subset \mathbb{R}$, where $\sigma(A)$ stands for the spectrum of A, and there exists an invertible matrix D such that

$$A = D^{-1} J_A D,$$

where

$$J_A = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & \lambda_{n-1} & 0 \\ 0 & 0 & \dots & 0 & \lambda_n \end{pmatrix}.$$

Now we will show that

$$\sigma(A) = \{1\}.$$

Let $\lambda \in \sigma(A)$. Then then there exists a non-trivial eigenvector $v \in \mathbb{C}^n$ such that

$$Av = \lambda v. \tag{27.1}$$

On the other hand, since $w(A) = 0$, one gets

$$v^\top (A^5 - A^4 + 2A^3 - 2A^2 + A)v = v^\top I v. \tag{27.2}$$

By applying (27.1) in (27.2), one can conclude that

$$v^\top (A^5 - A^4 + 2A^3 - 2A^2 + A)v = (\lambda^5 - \lambda^4 + 2\lambda^3 - 2\lambda^2 + \lambda)|v|^2$$

and $v^\top I v = |v|^2$. Hence (27.2) takes the following equivalent form:

$$|v|^2 w(\lambda) = 0 \iff w(\lambda) = 0.$$

Thus

$$\lambda \in \sigma(A) \Longrightarrow w(\lambda) = 0.$$

But $\sigma(A) \subset \mathbb{R}$ and hence $\sigma(A) = \{1\}$, because $w(x) = 0$ has only one real root. Thus

$$J_A = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} = I.$$

Finally, one has

$$A = D^{-1} J_A D = D^{-1} I D = I,$$

which completes the solution. $\square$

Problem P

Assume that we are given two football players P_1 and P_2. Let us denote by C_i the average of scored goals per match for P_i while playing for a club and N_i the average of scored goals per match scored for the national team. Finally for both players P_i, we denote by T_i the average of scored goals per match regardless whether the match was for the club or the national team. We tacitly assume here that both players played for a club and also the national team. Is it possible that $C_1 < C_2$ and $N_1 < N_2$, but $T_2 < T_1$?

Solution:

Yes, it is possible. Let us denote by CM_i the club matches of player P_i and by NM_i the national team matches for player P_i. Furthermore, we let CS_i the scored goals in club matches by player P_i and NS_i the corresponding goals for the national team. We need

$$\frac{\mathrm{CS}_1}{\mathrm{CM}_1} < \frac{\mathrm{CS}_2}{\mathrm{CM}_2}, \; \frac{\mathrm{NS}_1}{\mathrm{NM}_1} < \frac{\mathrm{NS}_2}{\mathrm{NM}_2}, \text{ and } \frac{\mathrm{CS}_2 + \mathrm{NS}_2}{\mathrm{CM}_2 + \mathrm{NM}_2} < \frac{\mathrm{CS}_1 + \mathrm{NS}_1}{\mathrm{CM}_1 + \mathrm{NM}_1}.$$

Let us assume that scoring for the national team is much harder than for the club. Hence, if P_1 did not play much for the national team, then T_1 will be near the average for the club and thus quite high. On the other hand, if P_2 played a lot for the national team, then T_2 will be near this average.

To make the problem easier, we assume that both players played the same amount of matches. Furthermore, the problem is scale invariant, hence we may assume that they played 100 matches. Let us go for a very extreme situation. We assume that $\mathrm{NM}_1 = 1$ and $\mathrm{NS}_1 = 0$ and $\mathrm{NM}_2 = 99$ as well as $\mathrm{NS}_2 = 1$. Then clearly P_2 has the better average for matches with the national team. Now, we know that $\mathrm{CM}_2 = 1$, and again going for the extreme: $\mathrm{CS}_2 = 1$. Furthermore, $\mathrm{CM}_1 = 99$. Going for the extreme: $\mathrm{CS}_1 = 98$. Although P_1 scored a lot while playing for a club, its average is still worse than that of P_2. Finally, P_1 has scored 99 times in 100 matches, while player P_2 only scored twice. Hence, P_1 has the much better average. $\square$

Problem P.1
Two candidates, Aiden and Bump, are running for president. The vote, in which $2n$ valid votes were cast (a valid vote is a vote cast for one of the candidates), ended in a tie. What is the probability that the leading person changed only once during the vote count (a tie does not count as a change of lead).

Solution:
We can equate the situation during the vote count with a path on a grid $\mathbb{Z}^2$. Namely, consider the sequence of points $(A_k)_{k=0}^n$, where $A_k = (k, a_k)$, where a_k is equal to the difference (among first k counted votes) in the number of votes cast for Aiden minus the votes cast for Bump. The number of all such paths is equal to $\binom{2n}{n}$.
Now, to calculate the probability of the event we are interested in, we can, for reasons of symmetry, consider the situation when the candidate A (i.e. Aiden) first started to lead, and then at some point the candidate B (i.e. Bump) gained the advantage. The final probability will be equal to the doubled result.
We have $a_0 = 0$,and $a_{k+1} = a_k + 1$ if the k-th vote was for Aiden and $a_{k+1} = a_k - 1$ if the k-th vote was for Bump. Since the first vote was for Aiden and the last for Bump, we are only interested in the paths that pass through points $(1, 1)$ and $(2n - 1, -1)$. The number of such paths is equal to $\binom{2n-2}{n}$.
However, we are only interested in paths that cross the OX axis only once. Therefore, we must subtract from the above value the number of paths that cross the OX axis at least twice. Such paths first reach level -1, then return to level 1 to go down to level -1 again. Let k_1 be the first moment to reach level -1 (i.e. $k_1 = \min\{k \mid a_k = -1\}$), and k_2 – the first moment of returning to level 1 (i.e. $k_2 = \min\{k \mid a_k = 1, k > k_1\}$). Such a path can now be identified with the path $(\hat{A}_k)_{k=0}^n$ from $(1, -3)$ to $(2n - 1, 3)$ by symmetric reflection with respect to the line $y = -1$ of the initial part of the path, and the last part with respect to the line $y = 1$. Namely, $\hat{A}_k = (k, \hat{a}_k)$, where $\hat{a}_k = -2 - a_k$ for $k = 1, \ldots, k_1$, $\hat{a}_k = a_k$ for $k = k_1, \ldots, k_2$, and $\hat{a}_k = 2 - a_k$ for $k = k_2, \ldots, 2n - 1$.
Such an identification is mutually unequivocal, i.e. we can transform any path from $(1, -3)$ to $(2n - 1, 3)$ into a path that intersects the OX axis at least twice. Thus the number of bad paths is equal to $\binom{2n-2}{n+2}$. Subtracting above values gives the final count of paths that cross the OX axis only once, i.e.

$$\binom{2n-2}{n} - \binom{2n-2}{n+2} = \frac{4\,(2n-1)!}{(n+2)!\,(n-2)!} = \frac{4}{n+2}\binom{2n-1}{n+1}.$$

Hence the probability asked in the problem equals to

$$2\frac{\frac{4}{n+2}\binom{2n-1}{n+1}}{\binom{2n}{n}} = \frac{4(n-1)}{(n+1)(n+2)}.$$

□

Solution no.2:
Such paths presented in the first solution are related to the Catalan numbers.

Lemma 1
The number of paths from $(0, 0)$ to $(2n, 0)$ that are contained in the upper half-plane (i.e. for which $a_k \geqslant 0$ for all $k = 0, 1, \ldots, 2n$) is equal to the n-th Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Proof: Let us now consider a path of length $2n$ and let $2j$ be the first point of return to 0. We can identify such a path as the union of two shorter paths: the first from $(1,1)$ to $(2j-1,1)$ and the second from $(2j,0)$ to $(2n,0)$. The number of such paths is thus $C_{j-1}C_{n-j}$. Hence the number of all paths is equal to

$$C_n = \sum_{j=1}^{n} C_{j-1}C_{n-j}\,. \tag{29.1}$$

To obtain the exact formula for C_n we will use the method of generating function. So let $f(z) = \sum_{n=0}^{\infty} C_n z^n$. We have of course $C_n \leqslant \binom{2n}{n} < 2^{2n}$, hence the series converges for $|z| < \frac{1}{4}$. Equation (29.1) gives

$$\begin{aligned} f(z) - C_0 = \sum_{n=1}^{\infty} C_n z^n &= \sum_{n=1}^{\infty}\sum_{j=1}^{n} C_{j-1}C_{n-j} z^n \\ \sum_{m=0}^{\infty}\sum_{k=0}^{m} C_k z^k\, C_{m-k} z^m\, z &= z f(z)^2\,, \end{aligned}$$

where the second line is obtained by substituting $j = k+1$ and $n = m+1$. Hence the function f satisfies equation $zf(z)^2 - f(z) - 1 = 0$. Only one of its solutions is an analytic function around $z = 0$, namely $f(z) = \dfrac{1-\sqrt{1-4z}}{2z}$.
Considering the expansion of the square root

$$\begin{aligned} \sqrt{1-4z} &= \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-4z)^n = \sum_{n=0}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!}(-1)^n 2^{2n} z^n \\ &= \sum_{n=0}^{\infty} \frac{1\cdot 1\cdot 3\cdots(2n-3)}{n!}(-1)2^n z^n \end{aligned}$$

leads to

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \frac{1\cdot 3\cdots(2n-3)}{n!} 2^{n-1} z^{n-1} = \sum_{n=0}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{(n+1)!} 2^n z^n \\ &= \sum_{n=1}^{\infty} \frac{(2n)!}{(n+1)!\,n!} z^n\,. \end{aligned}$$

From the above expansion we get

$$C_n = \frac{(2n)!}{(n+1)!\,n!} = \frac{1}{n+1}\binom{2n}{n}\,.$$

◇

Let us now count in the same way all paths from $(0,0)$ to $(2n,0)$ that cross the OX axis only once. Let $(2k,0)$ be the crossing point for the specific path. The number of such paths (i.e. crossing the OX axis at $(2k,0)$) is thus $C_k C_{n-k}$, hence the number of all paths is equal to

$$\sum_{k=1}^{n-1} C_k C_{n-k} = \sum_{j=2}^{n} C_{j-1}C_{n+1-j} = C_{n+1} - 2C_0C_n = \frac{4}{n+2}\binom{2n-1}{n+1}\,.$$

The rest is as in the first solution. □

Problem P.2

Numbers $0, 1, 2, \ldots, 2021$ are distributed randomly on a circle. We start from 2021 and go along the circle. Every encountered number which is smaller then all other numbers is removed. (So we first remove 0, then 1, and so on up to 2021.) We stop after removing all numbers. What is the expected value of rotation along the circle to be made?

Solution:

We will do more generally for n points.

Let X_n be the random variable describing number of rotations and

$$A = \{\text{numbers } n-2, n-1, n \text{ are distributed in the orientation of the walk}\}.$$

$P(A) = \frac{1}{2}$. Moreover $E(X_n|A) = E(X_{n-1})$ and $E(X_n|A') = E(X_{n-1}) + 1$, so

$$E(X_n) = E(X_n|A)P(A) + E(X_n|A')P(A') = E(X_{n-1}) + \frac{1}{2}.$$

Obviously $E(X_1) = 1$, which easily gives $E(X_n) = \frac{n+1}{2}$.

So in our problem we make 1011 rotations on average. □

Solution no.2:

Let X_i be the random variable describing position of the number i on the circle, or – more precisely – describing what part of the circle must be traversed from the origin to the point i. Hence, we can consider X_i as variables uniformly distributed on the interval $[0, 1)$. The distance (in the part of the rotation of the circle) that we have to cover from the point X_{i-1} to the next point X_i is described by the function

$$d(X_{i-1}, X_i) = \begin{cases} X_i - X_{i-1} & , X_i \geqslant X_{i-1} \\ 1 + X_i - X_{i-1} & , X_i < X_{i-1} \end{cases}$$

So the total distance to cover is equal to

$$\sum_{i=0}^{2021} d(X_{i-1}, X_i)\,,$$

where we put $X_{-1} = X_{2021}$ as starting point.

We have

$$E\Big(d(X_{i-1}, X_i)\Big) = \iint\limits_{[0,1]^2} d(x, y)\, dxdy = \frac{1}{2}\,.$$

Thus the expected value of rotations are equal to $(2021 + 1) \cdot \frac{1}{2} = 1011$. □