List of Problems

for the

Individual Part

ISTCiM 2022

Individual Part

Problem A.
Let $A_1, \dots, A_{2022}$ be subsets of $S = \{1, 2, \dots, 1011\}$ such that each set A_i has 11 elements and each element in S is in exactly m sets A_i. Find m.

Problem C.
Show that

$$\int_0^\infty \frac{e^{-tx}}{1+x^2}\mathrm{d}x = \int_0^\infty \frac{\sin x}{t+x}\mathrm{d}x$$

for every $t > 0$.

Problem E.
Let $a_1, \dots, a_n$ be any positive real numbers. Show that

$$\sqrt[n]{a_1^n} + \sqrt[n]{a_1^n + a_2^n} + \cdots + \sqrt[n]{a_1^n + \cdots + a_n^n} \geqslant \sqrt[n]{a_1^n + (a_1 + a_2)^n + \cdots + (a_1 + \cdots + a_n)^n}\,.$$

When is equality achieved?

Problem G.
Let $n \geqslant 3$ be an integer and let $P_1, P_2, \dots, P_n$ be distinct points in the plane, no three of them collinear. Show that there is an angle $\sphericalangle P_i P_j P_k$ (for distinct i, j, k), which is less or equal to $\frac{\pi}{n}$.

Problem P.
A certain type of bacteria splits every second either into two perfect copies of itself or disintegrates. The probability of splitting is equal to p, and all copies are independent.
What is the probability that one bacterium will produce an everlasting colony (i.e. the probability that a colony arising from a single bacterium will never die out)?

Problem A.
Let $A_1, \ldots, A_{2022}$ be subsets of $S = \{1, 2, \ldots, 1011\}$ such that each set A_i has 11 elements and each element in S is in exactly m sets A_i. Find m.

Solution:
Let M be the incidence 2022×1011 matrix, where rows represent the subsets $A_1, \ldots, A_{2022}$ and columns represent elements of S. It means that the $m_{i,j}$ element of M in the i-th row and j-th column equals 1 if and only if the j-th element of S belongs to A_i' otherwise the $m_{i,j} = 0$. Since every set A_i contains 11 elements, each row contains eleven 1, so the total number of ones in our matrix is $11 \cdot 2022$. On the other hand each element of S belongs to exactly m sets A_i. Thus each column of our matrix contains m ones. Hence the total number of ones in the matrix is $m \cdot 1011$ as there are 1011 columns. Thus we have $m \cdot 1011 = 11 \cdot 2022$ so $m = 22$. $\square$

Problem C.
Show that

$$\int_0^\infty \frac{e^{-tx}}{1+x^2}\mathrm{d}x = \int_0^\infty \frac{\sin x}{t+x}\mathrm{d}x$$

for every $t > 0$.

Solution:
Let

$$f(t) = \int_0^\infty \frac{e^{-tx}}{1+x^2}\mathrm{d}x \quad \text{and} \quad g(t) = \int_0^\infty \frac{\sin x}{t+x}\mathrm{d}x\,.$$

We will show that both functions f and g satisfy the equation $y''(t) + y(t) = \frac{1}{t}$ with the same boundary conditions.
As we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\frac{e^{-tx}}{1+x^2}\right| = \frac{xe^{-tx}}{1+x^2} \leqslant e^{-xt},$$
$$\left|\frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{e^{-tx}}{1+x^2}\right| = \frac{x^2e^{-tx}}{1+x^2} \leqslant e^{-tx},$$
$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\frac{\sin x}{t+x}\right| = \frac{|\sin x|}{(t+x)^2} \leqslant \frac{1}{(t+x)^2},$$
$$\left|\frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\sin x}{t+x}\right| = 2\frac{|\sin x|}{(t+x)^3} \leqslant \frac{2}{(t+x)^3},$$

both integrals can be differentiated under the integral sign by the Leibniz integral rule for Lebesgue integral. Hence

$$f'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \frac{e^{-tx}}{1+x^2}\mathrm{d}x = \int_0^\infty \frac{-xe^{-tx}}{1+x^2}\mathrm{d}x\,,$$
$$f''(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \frac{-xe^{-tx}}{1+x^2}\mathrm{d}x = \int_0^\infty \frac{x^2e^{-tx}}{1+x^2}\mathrm{d}x$$
$$= \int_0^\infty \frac{1+x^2-1}{1+x^2}e^{-tx}\mathrm{d}x = \int_0^\infty e^{-tx}\mathrm{d}x - \int_0^\infty \frac{e^{-tx}}{1+x^2}\mathrm{d}x$$
$$= \frac{1}{t} - f(t)$$

and

$$g'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \frac{\sin x}{t+x}\mathrm{d}x = -\int_0^\infty \frac{\sin x}{(t+x)^2}\mathrm{d}x\,,$$
$$g''(t) = 2\int_0^\infty \frac{\sin x}{(t+x)^3}\mathrm{d}x = -\frac{\sin x}{(t+x)^2}\bigg|_{x=0}^{x=\infty} + \int_0^\infty \frac{\cos x}{(t+x)^2}\mathrm{d}x$$
$$= -\frac{\cos x}{t+x}\bigg|_{x=0}^{x=\infty} - \int_0^\infty \frac{\sin x}{t+x}\mathrm{d}x = \frac{1}{t} - g(t)\,,$$

where we integrated by parts in calculation of the second derivative. Hence the difference $f - g$ is the solution of the well-known differential equation $y'' + y = 0$. Thus

$$f(t) - g(t) = A\cos t + B\sin t$$

for some real constants A and B.
On the other hand both integrals tends to 0 as t tends to infinity, so

$$\lim_{t\to+\infty}\big(f(t) - g(t)\big) = \lim_{t\to+\infty} f(t) - \lim_{t\to+\infty} g(t) = 0\,,$$

which implies $A = B = 0$, and therefore $f(t) = g(t)$ as desired. $\square$

Problem E.
Let $a_1, \dots, a_n$ be any positive real numbers. Show that

$$\sqrt[n]{a_1^n} + \sqrt[n]{a_1^n + a_2^n} + \cdots + \sqrt[n]{a_1^n + \cdots + a_n^n} \geqslant \sqrt[n]{a_1^n + (a_1 + a_2)^n + \cdots + (a_1 + \cdots + a_n)^n}\,.$$

When is equality achieved?

Solution:
Let us fix $n \in \mathbb{N}$ and let V_n be a linear space $\mathbb{R}^n$ equiped with the norm $\|\mathrm{v}\| = \sqrt[n]{|v_1|^n + \cdots + |v_n|^n}$, where $\mathrm{v} = (v_1, \dots, v_n) \in V_n$. Let us now consider vectors $\mathrm{v}_1, \dots, \mathrm{v}_n \in V_n$ given by

$$\begin{aligned} \mathrm{v}_1 &= (a_1, 0, 0, \dots, 0), \\ \mathrm{v}_2 &= (a_2, a_1, 0, \dots, 0), \\ &\vdots \\ \mathrm{v}_n &= (a_n, a_{n-1}, a_{n-2}, \dots, a_1). \end{aligned}$$

Then the inequality to be proved becomes the triangle inequality

$$\|\mathrm{v}_1\| + \cdots + \|\mathrm{v}_n\| \geqslant \|\mathrm{v}_1 + \cdots + \mathrm{v}_n\|.$$

Equality can be only achieved when all vectors $\mathrm{v}_1, \dots, \mathrm{v}_n$ are collinear, and this is only possible when all but the last are equal to zero. This means that equality is achieved if and only if $a_1 = \cdots = a_{n-1} = 0$. □

Problem G.
Let $n \geqslant 3$ be an integer and let $P_1, P_2, \ldots, P_n$ be distinct points in the plane, no three of them collinear. Show that there is an angle $\sphericalangle P_i P_j P_k$ (for distinct i, j, k), which is less or equal to $\dfrac{\pi}{n}$.

Solution:
The convex hull of points $P_1, P_2, \ldots, P_n$ is a convex k-gon, whose vertices are some of given points $P_1, P_2, \ldots, P_n$ (they can be all of them), hence $k \leqslant n$. The sum of all vertex angles is $(k-2)\pi$. By the Pigeonhole principle[1] there is a vertex whose angle is less or equal to $\frac{k-2}{k}\pi = \left(1 - \frac{2}{k}\right)\pi \leqslant \left(1 - \frac{2}{n}\right)\pi = \frac{n-2}{n}\pi$. Let us now draw lines from this vertex to all other points. These lines form $n-2$ angles, whose sum is the vertex angle. Once again by the Pigeonhole principle one of such angles is less than $\frac{1}{n-2}\left(\frac{k-2}{k}\pi\right) \leqslant \frac{1}{n-2}\frac{n-2}{n}\pi = \frac{\pi}{n}$, which had to be demonstrated. □

[1] Other common names are *Dirichlet's drawer principle* or *Schubfachprinzip*.

Problem P.
A certain type of bacteria splits every second either into two perfect copies of itself or disintegrates. The probability of splitting is equal to p, and all copies are independent.
What is the probability that one bacterium will produce an everlasting colony (i.e. the probability that a colony arising from a single bacterium will never die out)?

Solution 1.:
Let P be the probability that the initial bacterium will generate a perpetual colony. We have

$$P = p(1 - (1 - P)^2)\,, \tag{10.1}$$

because the first bacterium must divide at the start and then at least one of its descendants must generate an everlasting colony. The equation (10.1) has two solutions: $P_1 = 2 - \frac{1}{p}$ and $P_2 = 0$.
If $p \leqslant \frac{1}{2}$ then the unique admissible solution is $P = 0$.
If $p > \frac{1}{2}$ we will proceed as follows. Let $f(x) = p(1 - (1 - x)^2) = px(2 - x)$. Since function f describes a probabilistic process, the value of the derivative $f'(x_0)$ at the fixed point x_0 of the function f (i.e. $x_0 = f(x_0)$) will show whether x_0 is admissible, and more specifically whether x_0 is a stable, attracting fixpoint. This is the case when the absolute value of the derivative $f'(x_0)$ is less than 1. If it is greater, the fixed point is repulsive, so it cannot be a limit point of a probabilistic process.
In our case we have $f'(x) = 2p - 2px = 2p(1 - x)$, so $f'(P_1) = 2p\big(1 - (2 - \frac{1}{p})\big) = 2(1 - p) < 1$ and $f'(P_2) = 2p > 1$ (where $p > \frac{1}{2}$). Hence the correct and admissible solution is $P = 2 - \frac{1}{p}$. $\square$

Solution 2.:
Let P_n be the probability that a colony generated by one initial bacterium lasts at least n seconds. The sequence (P_n) is decreasing (because the n-th generation requires all previous generations to exist). We also, of course, have $P_0 = 1$ and $P_1 = p$. Furthermore, we have the recursion

$$P_{n+1} = p(1 - (1 - P_n)^2)\,, \tag{10.2}$$

because the first bacterium must divide at the start, and then one of its descendants must survive at least n generations. As the sequence (P_n) is decreasing and bounded from below (by 0), it converges to the probability in question, which satisfy the equation

$$P = p(1 - (1 - P)^2) = pP(2 - P)\,.$$

The equation has two solutions: 0 and $2 - \frac{1}{p}$. If $p \leqslant \frac{1}{2}$ then $2 - \frac{1}{2} \leqslant 0$, so the unique possible solution is $P = 0$. If instead $p > \frac{1}{2}$, then if $P_n > 2 - \frac{1}{p}$ we have $(1 - P_n)^2 < (\frac{1}{p} - 1)^2$, hence

$$P_{n+1} = p\big(1 - (1 - P_n)^2\big) > 2\Big(1 - \big(\frac{1}{p} - 1\big)^2\Big) = 2 - \frac{1}{p}\,.$$

Hence by induction on n, as $P_0 = 1 > 2 - \frac{1}{p}$, we get $P_n > 2 - \frac{1}{p}$ for all n. Thus $\lim\limits_{n\to\infty} P_n \geqslant 2 - \frac{1}{p}$, so in this case the solution must be $P = 2 - \frac{1}{p}$. $\square$

Team Part

Problem A.1.
Let n be a positive integer. Find the number of different solutions of the equation $x^2 \equiv x \pmod{n}$ in the ring $\mathbb{Z}_n$ of integers modulo n.

Problem A.2.
A family $\{A_\tau\}_{\tau\in\mathcal{I}}$ consisting of at least two sets is called a partition of a given set X, if all sets A_τ are pairwise disjoint and $\bigcup_{\tau\in\mathcal{I}} A_\tau = X$.

(a) Construct a partition of the set of all positive integers into an infinite number of disjoint infinite arithmetic progressions with different common differences.

(b) Can the set of all positive integers be partitioned into a finite number of such progressions?

Problem C.1.
Show that both $\cos\alpha$ and $\sin\alpha$ are rational if and only if either $\tan\frac{\alpha}{2}$ is rational or α is an odd multiply of π.

Problem C.2.
Let a continuous function $f\colon (0,+\infty) \to \mathbb{R}$ satisfy $\lim_{n\to\infty} f(n\alpha) = 0$ for all $\alpha > 0$. Must $f(x)$ tend to 0 as $x \to +\infty$?

Problem E.1.
Let A be a positive-definite Hermitian $n\times n$ matrix, with λ and μ its minimal and maximal eigenvalues. Prove that

$$\det\left(\frac{1}{\mu}A + \mu A^{-1}\right) \geqslant \left(\frac{\operatorname{tr} A}{n\mu} + \frac{n\mu}{\operatorname{tr} A}\right)^n$$
$$\det\left(\frac{1}{\lambda}A + \lambda A^{-1}\right) \geqslant \left(\frac{n}{\lambda \operatorname{tr}(A^{-1})} + \frac{\lambda \operatorname{tr}(A^{-1})}{n}\right)^n .$$

Problem E.2.
Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x)f(x+y) \geqslant f(x)^2 + xy$$

for all $x, y \in \mathbb{R}$.

Problem G.1.
Let each of the three main diagonals of a hexagon halves it (i.e. divides the hexagon into two parts with equal areas). Show that these diagonals are concurrent (i.e. they have a point in common).

Problem G.2.

Let $A = \begin{pmatrix} p & q & r \\ r & p & q \\ q & r & p \end{pmatrix}$, where $p, q, r > 0$ and $p + q + r = 1$. Find $\lim_{n\to\infty} A^n$.

Problem P.1.
We assume that $k, n \in \mathbb{N}$, $k < n$. Let A be the event that in n tosses of a fair coin there are at least k consecutive heads and let B be the event that in n tosses of a fair coin there are at least either $k+1$ consecutive heads or $k+1$ consecutive tails. Which of the events has greater probability?

Problem P.2.
Is there a subset $A \subset [0,1]$ such that A and $[0,1] \setminus A$ are homeomorphic?
Remark
Two metric spaces X and Y are said to be homeomorphic if there is a bijection $\phi\colon X \to Y$ such that both ϕ and ϕ^{-1} are continuous on X and Y respectively.

Problem A.1.
Let n be a positive integer. Find the number of different solutions of the equation $x^2 \equiv x \pmod{n}$ in the ring $\mathbb{Z}_n$ of integers modulo n.

Solution:
The number of distinct solutions is equal to 2^{k_n}, where k_n is the number of distinct prime divisors of n, i.e. where $n = p_1^{i_1} \cdots p_{k_n}^{i_{k_n}}$ and all p_i are different. The equation $x^2 \equiv x \pmod{n}$ is equivalent to the system of equations $\forall_j\ x^2 - x = x(x-1) \equiv 0 \pmod{p_j^{k_j}}$. Since the gratest common divisor of x and $x-1$ is one, each congruence $x(x-1) \equiv 0 \pmod{p_j^{k_j}}$ has only two distinct solutions, $x \equiv 0$ and $x \equiv 1 \pmod{p_j^{k_j}}$. Thus by the Chinese Remainder Theorem there are 2^{k_n} distinct solutions to the given congruence. $\square$

Problem A.2.
A family $\{A_\tau\}_{\tau\in\mathcal{I}}$ consisting of at least two sets is called a partition of a given set X, if all sets A_τ are pairwise disjoint and $\bigcup\limits_{\tau\in\mathcal{I}} A_\tau = X$.

(a) Construct a partition of the set of all positive integers into an infinite number of disjoint infinite arithmetic progressions with different common differences.

(b) Can the set of all positive integers be partitioned into a finite number of such progressions?

Solution:
Ad (a). Let $A(\alpha,\delta)$ denote an arithmetic progression $A(\alpha,\delta) = \{\alpha,\ \alpha{+}\delta,\ \alpha{+}2\delta, \dots,\ \alpha{+}n\delta, \dots\}$. We can represent a positive integer as the product of an odd number and a power of two in the unique way as $2^n(1+2m)$, which gives us the desired partitioning

$$\mathbb{N} = \Big\{2^n(1+2m)\ :\ m,n\in\mathbb{N}\Big\} = \bigcup_{n=0}^{\infty}\Big\{2^n + 2^{n+1}m\ :\ m\in\mathbb{N}\Big\} = \bigcup_{n=0}^{\infty} A\Big(2^n, 2^{n+1}\Big).$$

Ad (b). The answer is negative, i.e. the set of all positive integers cannot be partitioned into at least two but a finite number of pairwise disjoint arithmetic progressions with different common differences. We will proceed by contradiction.

Let $\mathbb{N} = \bigcup\limits_{k=1}^{N} A(a_k, d_k)$, where $N \geqslant 2$, and let us assign to each progression A_k its generating series $\sum\limits_{n=0}^{\infty} z^{a_k+nd_k}$. Note that the series converges absolutely to $f_k(z) = \dfrac{z^{a_k}}{1-z^{d_k}}$ for $|z|<1$. Now, since the sets $A_1,\dots,A_N$ partition $\mathbb{N} = \{1,2,\dots\}$, the sum of their corresponding generating series equals to the generating series of positive integers, i.e.

$$f_1(z) + \cdots + f_N(z) = \sum_{n=1}^{\infty} z^n = \frac{z}{1-z},$$

hence

$$\frac{z^{a_N}}{1-z^{d_N}} = \frac{z}{1-z} - \sum_{k=1}^{N-1}\frac{z^{a_k}}{1-z^{d_k}}. \tag{22.1}$$

Without loss of generality we can assume that $1 < d_1 < d_2 < \cdots < d_N$. Passing to the limit as z tends to $\xi = e^{2\pi i/d_N}$ (i.e. to the primitive d_N-th root of unity) gives the contradiction, as the right-hand side of (22.1) converges to $\dfrac{\xi}{1-\xi} - \sum\limits_{k=1}^{N-1}\dfrac{\xi^{a_k}}{1-\xi^{d_k}}$ (because $\xi^\delta \neq 1$ for $0<\delta<1$), and its left-hand side simultaneously diverges to the infinity. □

Problem C.1.
Show that both $\cos\alpha$ and $\sin\alpha$ are rational if and only if either $\tan\frac{\alpha}{2}$ is rational or α is an odd multiply of π.

Solution 1.:
The equivalence follows from a simple geometrical fact. Let $O = (0,0)$, $A = (-1,0)$, $B = (1,0)$ and $C = (\cos\alpha, \sin\alpha)$. Of course the angle $\sphericalangle BOC = \alpha$. Let us assume that α is not an odd multiply of π, i.e. points A and C are distinct. Otherwise $\cos\alpha = -1$ and $\sin\alpha = 0$, which are both rational.
All points A, B, C lie on the unit circle, hence $\sphericalangle BAC = \frac{1}{2}\sphericalangle BOC = \frac{1}{2}\alpha$. Thus the slope of the line AC is equal to $\tan\frac{\alpha}{2}$. On the other hand, as triangle $\triangle ABC$ is a right triangle, the line BC is perpendicular to AC, hence its slope is equal to $-\left(\tan\frac{\alpha}{2}\right)^{-1}$.
Now if the $\tan\frac{\alpha}{2}$ is rational then the point of intersection of the lines AC and BC (i.e. point C) has rational coordinates (being the solution of system of linear equations with rational coefficients).
Conversely, if point C has rational coordinates, the slope of the line AC is also rational, as it passes through two points (i.e. A and C), whose coordinates are rational. □

Solution 2.:
The angle α is an odd multiply of π if and only if $\cos\frac{\alpha}{2} = 0$. In that case $\cos\alpha = -1$ and $\sin\alpha = 0$ (which are both rational), and vice-versa. So let us assume now that α is not an odd multiply of π.

Implication ($\Rightarrow$).
We have
$$\sin\alpha = 2\,\sin\frac{\alpha}{2}\,\cos\frac{\alpha}{2} \tag{23.1}$$
and
$$1 + \cos\alpha = 2\,\cos^2\frac{\alpha}{2}\,.$$
Dividing above equations (under the assumption that $\cos\frac{\alpha}{2} \neq 0$) gives
$$\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}\,.$$
which means that the $\tan\frac{\alpha}{2}$ is a rational number as long as $\cos\alpha$ and $\sin\alpha$ are rational.

Implication ($\Leftarrow$).
In addition to (23.1) we have also
$$\cos\alpha = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\,.$$
Using the trigonometric identity $1 = \cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}$ leads (under the assumption that $\cos\frac{\alpha}{2} \neq 0$) to
$$\sin\alpha = \frac{2\,\sin\frac{\alpha}{2}\,\cos\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}} = \frac{2\,\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$$
and
$$\cos\alpha = \frac{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}} = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}\,.$$
which shows that $\cos\alpha$ and $\sin\alpha$ are rational as long as $\tan\frac{\alpha}{2}$ is a rational number. □

Problem C.2.
Let a continuous function $f\colon (0,+\infty) \to \mathbb{R}$ satisfy $\lim\limits_{n\to\infty} f(n\alpha) = 0$ for all $\alpha > 0$. Must $f(x)$ tend to 0 as $x \to +\infty$?

Solution:
Yes, it must, i.e. $\lim\limits_{x\to+\infty} f(x) = 0$. To show it we will need

Baire's Category Theorem
A non-empty complete metric space is not a set of the first category (i.e. meager set). That is, it is not a countable union of nowhere dense sets.

Corollary
If $\mathbb{R} = \bigcup\limits_{n=1}^{\infty} F_n$*, where all* F_n *are closed, then at least one* F_{n_0} *must contain a non-empty interval.*

The above theorem (and its corollary) is in a standard course of real analysis, so we cite it without proof.
Let now $\varepsilon > 0$ and let

$$F_k = \left\{ a > 0 \ : \ \bigl|f(na)\bigr| \leqslant \frac{1}{2}\varepsilon \text{ for all } n \geqslant k \right\}.$$

All sets F_k are closed as the function f is continuous. Moreover $F_n \subset F_{n+1}$, and $\bigcup\limits_{n=1}^{\infty} F_n = \mathbb{R}$ by assumptions on the function f. Hence, the corollary of Baire's Category Theorem shows that there exist an n_0 and a non-empty interval $\bigl(a_0, b_0\bigr)$ such that $\bigl(a_0, b_0\bigr) \subset F_{n_0}$. It follows from properties of the set F_{n_0} that $\bigl|f(x)\bigr| \leqslant \frac{1}{2}\varepsilon$ for $x \in \bigl(ma_0, mb_0\bigr)$, where $m = n_0, n_0{+}1, n_0{+}2, \ldots$
Let now $N = \left\lfloor \dfrac{a_0}{b_0 - a_0} \right\rfloor > 0$. We have $\bigl(ma_0, mb_0\bigr) \cap \bigl((m+1)a_0, (m+1)b_0\bigr) \neq \varnothing$ for all $m > N$, as $mb_0 > (m+1)a_0$ for such m. Hence $\bigl|f(x)\bigr| \leqslant \frac{1}{2}\varepsilon < \varepsilon$ for all $x > \max(n_0, N+1) \cdot a_0$. Since the initial choice of epsilon was arbitrary, we get $\lim\limits_{x\to+\infty} f(x) = 0$. $\square$

Problem E.1.
Let A be a positive-definite Hermitian $n \times n$ matrix, with λ and μ its minimal and maximal eigenvalues. Prove that

$$\det\left(\frac{1}{\mu}A + \mu A^{-1}\right) \geqslant \left(\frac{\operatorname{tr} A}{n\mu} + \frac{n\mu}{\operatorname{tr} A}\right)^n$$
$$\det\left(\frac{1}{\lambda}A + \lambda A^{-1}\right) \geqslant \left(\frac{n}{\lambda \operatorname{tr}(A^{-1})} + \frac{\lambda \operatorname{tr}(A^{-1})}{n}\right)^n .$$

Solution:
Any Hermitian matrix A can be diagonalized by a unitary matrix, i.e. there exist matrices: unitary U and diagonal D with $\lambda_1, \ldots, \lambda_n$ on the main diagonal, such that $A = UDU^{-1}$. Note that $0 < \lambda = \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n = \mu$, because A is positive-definite.
If $t > 0$ then

$$\det\left(\frac{1}{t}A + tA^{-1}\right) = \det\left(U\left(\frac{1}{t}D + tD^{-1}\right)U^{-1}\right) = \det\left(\frac{1}{t}D + tD^{-1}\right) = \prod_{k=1}^{n}\left(\frac{\lambda_k}{t} + \frac{t}{\lambda_k}\right).$$

Let $f(x) = \ln\left(\frac{x}{t} + \frac{t}{x}\right)$. As the second derivative

$$f''(x) = \frac{\left(t^2 + (2+\sqrt{5}x^2\right)\left(t^2 + (2-\sqrt{5})x^2\right)}{x^2(x^2+t^2)^2} > 0$$

for $|x| < t\sqrt{2+\sqrt{5}}$, it is easy to see that f is convex in the interval $\left(0, \sqrt{2+\sqrt{5}}\,\right]$. Therefore if $\mu \leqslant t\sqrt{2+\sqrt{5}}$, then

$$\begin{aligned}\frac{1}{n}\ln\det\left(\frac{1}{t}A + tA^{-1}\right) &= \frac{1}{n}\sum_{k=1}^{n}\left(\frac{\lambda_k}{t} + \frac{t}{\lambda_k}\right) = \frac{1}{n}\sum_{k=1}^{n} f(\lambda_k)\\ &\geqslant f\left(\frac{1}{n}\sum_{n=1}^{n}\lambda_k\right) = f\left(\frac{\operatorname{tr} A}{n}\right) = \ln\left(\frac{\operatorname{tr} A}{nt} + \frac{nt}{\operatorname{tr} A}\right),\end{aligned}$$

which implies that

$$\det\left(\frac{1}{t}A + tA^{-1}\right) \geqslant \left(\frac{\operatorname{tr} A}{nt} + \frac{nt}{\operatorname{tr} A}\right).$$

Hence putting $t = \mu$ implies the first inequality (note that $\sqrt{2+\sqrt{5}} > 2$).

The second inquality follows from the first one by replacing A with A^{-1}, and μ with $\frac{1}{\lambda}$. $\square$

Problem E.2.
Find all functions $f\colon \mathbb{R}\to\mathbb{R}$ such that

$$f(x)f(x+y) \geqslant f(x)^2 + xy$$

for all $x, y \in \mathbb{R}$.

Solution:
All fuctions satisfying problem's conditions are:

$$\pm x,\ \pm|x|,\ \pm\sqrt{x^2+a^2} \ \text{ for } a\in\mathbb{R}\,.$$

Substituting $y := x$ in (26) gives

$$f(x)f(2x) \geqslant f(x)^2 + x^2 > 0, \tag{26.1}$$

hence $f(x) \neq 0$, for all $x \neq 0$. Then let us substitute $x := y$ and $y := x-y$ to get

$$f(y)f(x) \geqslant f(y)^2 + y(x-y)\,. \tag{26.2}$$

Hence we have

$$f(x)f(y) > 0$$

for both $x \geqslant y \geqslant 0$ and for $x \leqslant y \leqslant 0$. Hence the sign of $f(x)$ is constant for $x > 0$ and $x < 0$.

Let us now assume that $f(0) = 0$ and put $y := -x$ into (26) to get

$$0 = f(x)f(0) \geqslant f(x)^2 - x^2.$$

Hence $x^2 \geqslant f(x)^2$ for all $x \in \mathbb{R}$. Now using (26.1) with $(2x)^2 \geqslant f(2x)^2$ gives

$$4x^2 f(x)^2 \geqslant \big(f(2x)f(x)\big)^2 \geqslant \big(f(x)^2 + x^2\big)^2,$$

so $0 \geqslant \big(f(x)^2 - x^2\big)^2$, hence $f(x)^2 = x^2$ for all $x \in \mathbb{R}$. Since f doesn't change its sign in negative and in positive halfline, we get that $\pm x, \pm|x|$ are the only solutions satisfying $f(0) = 0$.

Now let us assume that $f(0) = a \neq 0$. It's easy to verify that if f is a solution, then $-f$ is also a solution (as well as $f(-x)$ and $-f(-x)$). Thus, without loss of generality we can assume that $a > 0$. Setting $x := 0$ and $y := x$ into (26) gives $a\,f(x) \geqslant a^2 > 0$, so $f(x) \geqslant f(0) = a > 0$ for all $x \in \mathbb{R}$. From (26.1) and *AM-GM* inequality we get $f(x)\,f(2x) \geqslant 2\,f(x)\,|x|$, so by substitution $2x := x$ we have $f(x) \geqslant |x|$ for all $x \in \mathbb{R}$. Now (26.2), with an eventual interchange of variables, gives

$$f(x) \geqslant f(y) + \frac{y(x-y)}{f(y)} \text{ and } f(y) \geqslant f(x) + \frac{x(y-x)}{f(x)},$$

hence

$$\frac{y(y-x)}{f(y)} \geqslant f(y) - f(x) \geqslant \frac{x(y-x)}{f(x)}. \tag{26.3}$$

Therefore

$$\big|y-x\big| \geqslant \max\left(\frac{|y|}{f(y)}, \frac{|x|}{f(x)}\right) \big|y-x\big| \geqslant \big|f(y)-f(x)\big|\,,$$

which shows that f is continuous. Moreover, from (26.3) we have $\dfrac{y}{f(y)} \geqslant \dfrac{f(y)-f(x)}{y-x} \geqslant \dfrac{x}{f(x)}$ for all $y > x$. Hence the function f is differentiable and $f'(x) = \dfrac{x}{f(x)}$. This implies that $\frac{\mathrm{d}}{\mathrm{d}x} f(x)^2 = 2x$, so $f(x)^2 = x^2 + a^2$, as $f(0) = a$. Hence $f(x) = \pm\sqrt{x^2+a^2}$, are the two solutions of (26) under the constraint $f(0) = \pm a$. $\square$

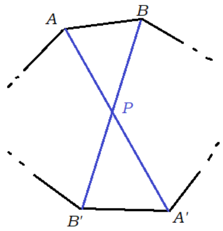
Problem G.1.
Let each of the three main diagonals of a hexagon halves it (i.e. divides the hexagon into two parts with equal areas). Show that these diagonals are concurrent (i.e. they have a point in common).

Solution:

Lemma 1

Let A, B and A', B' be two pairs of consecutive vertices of a given polygon, and let diagonals AA' and BB' divide the polygon into two parts with equal areas (cf. figure). Let P be the point of the intersection of these diagonals.
Then

$$\frac{PA}{PA'} = \frac{PB'}{PB}.$$

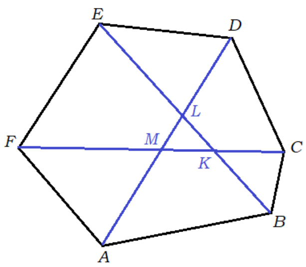


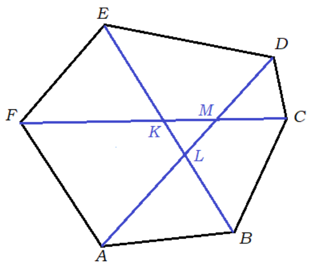
Proof: Diagonals $\overline{AA'}$ and $\overline{BB'}$ divide the polygon into parts of equal area, hence we have $S_{APB} + S_{APB'} = S_{A'PB} + S_{A'PB'}$ and $S_{APB} + S_{A'PB} = S_{APB'} + S_{A'PB'}$, where S_{XPY} denotes the area of the part of the polygon contained within the angle $\sphericalangle XPY$. Thus we get $S_{APB} = S_{A'PB'}$, which means that the areas of triangles $\triangle APB$ and $\triangle A'PB'$ are equal. And since the angles at the vertex P of both triangles are equal, the equality $PA \cdot PB = PA' \cdot PB'$ holds, i.e.

$$\frac{PA}{PA'} = \frac{PB'}{PB}.$$

◇

The main diagonals of the hexagon $ABCDEF$ either intersect at one point, or one of the following cases occurs (see figure below)





Without loss of generality we can assume the first case (the second can be solved in an analogical way). By lemma 1 we have

$$\frac{LA}{LD} = \frac{LE}{LB}, \qquad \frac{KB}{KE} = \frac{KF}{KC} \qquad \text{and} \qquad \frac{MC}{MF} = \frac{MA}{MD}.$$

On the other hand we have

$$\frac{LE}{LB} < \frac{KE}{KB}, \qquad \frac{KC}{KF} < \frac{MC}{MF} \qquad \text{and} \qquad \frac{MA}{MD} < \frac{LA}{LD}.$$

Thus we have a contradiction as

$$\frac{LA}{LD} = \frac{LE}{LB} < \frac{KE}{KB} = \frac{KC}{KF} < \frac{MC}{MF} = \frac{MA}{MD} < \frac{LA}{LD}.$$

Hence the only possible case is that the main diagonals concurre. □

Problem G.2.

Let $A = \begin{pmatrix} p & q & r \\ r & p & q \\ q & r & p \end{pmatrix}$, where $p, q, r > 0$ and $p + q + r = 1$. Find $\lim\limits_{n\to\infty} A^n$.

Solution:
Let $V = \mathbb{R}^3$ and β, γ be two functionals given by $\gamma(\mathrm{v}) = (v_x - v_y)^2 + (v_y - v_z)^2 + (v_z - v_x)^2$ and $\beta(\mathrm{v}) = v_x + v_y + v_z$ for $\mathrm{v} = (v_x, v_y, v_z) \in V$. We can see that $\gamma(\mathrm{v}) \geqslant 0$ and equality holds if and only if $v_x = v_y = v_z = \frac{1}{3}\beta(\mathrm{v})$. The functional γ can be expressed also as

$$\gamma(\mathrm{v}) = 2(v_x^2 + v_y^2 + v_z^2 - v_x v_y - v_y v_z - v_z v_x)\,.$$

Applying both functional to

$$A\mathrm{v} = \begin{pmatrix} pv_x + qv_y + rv_z \\ rv_x + pv_y + qv_z \\ qv_x + rv_y + pv_z \end{pmatrix},$$

gives $\beta(A\mathrm{v}) = (p+q+r)v_x + (q+p+r)v_y + (r+q+p)v_z = v_x + v_y + v_z = \beta(\mathrm{v})$ and

$$\begin{aligned}
\gamma(A\mathrm{v}) &= \big((p-q)v_x + (q-r)v_y + (r-q)v_z\big)^2 \\
&\quad + \big((p-q)v_x + (q-r)v_y + (r-q)v_z\big)^2 \\
&\quad + \big((p-q)v_x + (q-r)v_y + (r-q)v_z\big)^2 \\
&= \big((p-q)^2 + (q-r)^2 + (r-p)^2\big)(v_x^2 + v_y^2 + v_z^2) \\
&\quad + 2\big((p-q)(q-r) + (q-r)(r-p) + (r-p)(p-q)\big)(v_x v_y + v_y v_z + v_z v_x) \\
&= 2(p^2 + q^2 + r^2 - pq - qr - rp)(v_x^2 + v_y^2 + v_z^2 - v_x v_y - v_y v_z - v_z v_x) \\
&= c\gamma(\mathrm{v})\,,
\end{aligned}$$

where the constant c satisfies

$$c = p^2 + q^2 + r^2 - pq - qr - rp = \frac{1}{2}\big((p-q)^2 + (q-r)^2 + (r-p)^2\big) \geqslant 0$$

and

$$c = (p+q+r)^2 - 3(pq + qr + rp) = 1 - 3(pq + qr + rp) < 1\,,$$

hence $\gamma(A^n\mathrm{v}) = c^n\gamma(\mathrm{v}) \xrightarrow{n\to\infty} 0$. As $\beta(A^n\mathrm{v}) = \beta(\mathrm{v})$ we get

$$A^n\mathrm{v} \xrightarrow{n\to\infty} \frac{1}{3}\begin{pmatrix} \beta(\mathrm{v}) \\ \beta(\mathrm{v}) \\ \beta(\mathrm{v}) \end{pmatrix} = \frac{1}{3}\begin{pmatrix} v_x + v_y + v_z \\ v_x + v_y + v_z \\ v_x + v_y + v_z \end{pmatrix},$$

hence

$$\lim_{n\to\infty} A^n = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

□

Problem P.1.
We assume that $k, n \in \mathbb{N}$, $k < n$. Let A be the event that in n tosses of a fair coin there are at least k consecutive heads and let B be the event that in n tosses of a fair coin there are at least either $k+1$ consecutive heads or $k+1$ consecutive tails. Which of the events has greater probability?

Solution:
Let E_j be the subset of $\{H, T\}^j$ without k consecutive H for every $j \in \mathbb{N}$. Fix $j > k$. Every sequence in E_j may ends with terms $(T), (T, H), \dots, (T, \underbrace{H, \dots, H}_{k-1})$. Consequently

$$|E_j| = \sum_{m=j-k}^{j-1} |E_m|$$

for every $j > k$.
Let F_j be the subset of $\{H, T\}^j$ without $k+1$ consecutive H and without $k+1$ consecutive T for every $j \in \mathbb{N}$. Fix $j > k$. Every sequence in F_j may ends either with terms $(T, H), \dots, \underbrace{(T, H, \dots, H)}_{k+1}$ or with terms $(H, T), \dots, \underbrace{(H, T, \dots, T)}_{k+1}$. Let $a = (a_1, \dots, a_j) \in F_j$. We find the greatest $1 \leqslant m < j$ such that either $(a_m, a_{m+1}) = (T, H)$ or $(a_m, a_{m+1}) = (H, T)$. Then $(a_1, \dots, a_m) \in F_m$ and either $(a_{m+1}, \dots, a_j) = \underbrace{(H, \dots, H)}_{j-m}$ in the first case or $(a_{m+1}, \dots, a_j) = \underbrace{(T, \dots, T)}_{j-m}$ in the second case. Consequently

$$|F_j| = \sum_{m=j-k}^{j-1} |F_m|$$

for every $j > k$.
It is easy to check that $|E_m| = |F_m| = 2^m$ for every $1 \leqslant m \leqslant k-1$. Moreover $|E_k| = 2^k - 1$ and $|F_k| = 2^k$. Consequently

$$|F_j| = \sum_{m=j-k}^{j-1} |F_m| > \sum_{m=j-k}^{j-1} |E_m| = |E_j|$$

for every $j > k$. Therefore

$$P(A) = \frac{2^n - |E_n|}{2^n} > \frac{2^n - |F_n|}{2^n} = P(B).$$

$\square$

Problem P.2.
Is there a subset $A \subset [0,1]$ such that A and $[0,1] \setminus A$ are homeomorphic?
Remark
Two metric spaces X and Y are said to be homeomorphic if there is a bijection $\phi\colon X \to Y$ such that both ϕ and ϕ^{-1} are continuous on X and Y respectively.

Solution 1.:
The answer is positive. Let $\mathbb{R}^*$ denote the extended real line $[-\infty, +\infty]$. We have

$$[0,1] \simeq \mathbb{R}^*.$$

On the other hand

$$\mathbb{R}^* = \big\{-\infty\big\} \cup \bigcup_{n\in\mathbb{Z}} [n, n+1) \cup \big\{+\infty\big\}.$$

Let now $B = \big\{-\infty\big\} \cup \bigcup_{n\in\mathbb{Z}} \big[2n, 2n+1\big)$. We have $B \simeq \mathbb{R}^* \setminus B$ by the homeomorphism

$$\phi(x) = \begin{cases} +\infty, & \text{for } x = -\infty; \\ x - 4n - 1, & \text{for } x \in \big[2n, 2n+1\big). \end{cases}$$

The set A is the image of the set B in the homeomorphism between $\mathbb{R}^*$ and $[0,1]$. □

Solution 2.:
It is mostly the same solution as the first one above, but with the set A and the homeomorphism given in the explicite formulas.
As $\lim\limits_{x\to\infty} \dfrac{2^x}{2^x+1} = 1$ and $\lim\limits_{x\to-\infty} \dfrac{2^x}{2^x+1} = 0$ we have

$$\big[0,1\big] = \big\{0\big\} \cup \bigcup_{n\in\mathbb{Z}} \left[\frac{2^n}{2^n+1}, \frac{2^{n+1}}{2^{n+1}+1}\right) \cup \big\{1\big\}.$$

Let now

$$A = \big\{0\big\} \cup \bigcup_{n\in\mathbb{Z}} \left[\frac{2^{2n}}{2^{2n}+1}, \frac{2^{2n+1}}{2^{2n+1}+1}\right).$$

Hence

$$\big[0,1\big] \setminus A = \bigcup_{n\in\mathbb{Z}} \left[\frac{2^{2n-1}}{2^{2n-1}+1}, \frac{2^{2n}}{2^{2n}+1}\right) \cup \big\{1\big\}.$$

The homeomorphism between A and $\big[0,1\big] \setminus A$ is given by

$$\phi(x) = \begin{cases} 1, & \text{for } x = 0; \\ 0, & \text{for } x = 1; \\ x - \frac{2^{2n-1}-1}{(2^n+1)(2^{n-1}+1)}, & \text{for } \frac{2^n}{2^n+1} \leqslant x < \frac{2^{n+1}}{2^{n+1}+1}. \end{cases}$$

The function ϕ is clearly bijective and, restricted to either A or $\big[0,1\big] \setminus A$, continuous. □