ISTOCıM 2020
Problem list

Problem A.1
Let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ be the ring of integers modulo n. Find all pairs of non-empty subsets A and B such that $A \cup B = \mathbb{Z}_n$, $A \cap B = \varnothing$, $A + B \subset A$ and $A \cdot B \subset B$.
(We define $A + B = \{a + b \ : \ a \in A,\, b \in B\}$ and $A \cdot B = \{a \cdot b \ : \ a \in A,\, b \in B\}$, with all operations performed modulo n).

Problem A.2
Find all polynomials $P \in \mathbb{R}[x]$ (i.e. polynomials with real coefficients) such that $P(\mathbb{Q}) = \mathbb{Q}$.

Problem C.1
Suppose that $f\colon [0,1] \to \mathbb{R}$ is continuous on $[0,1]$ and differentiable on $(0,1)$ with $f(0) = 0$ and $f(1) = 1$. Prove that there exist 2020 distinct points $x_k \in (0,1)$ such that

$$\sum_{k=1}^{2020} \frac{1}{f'(x_k)} = 2020.$$

Problem C.2
Let $f\colon [0,1] \to [A,B]$ be a measurable function, where $A < B$ are positive numbers. Prove that

$$\int_0^1 f(x)\,dx - \left(\int_0^1 \frac{dx}{f(x)}\right)^{-1} \leqslant \left(\sqrt{B} - \sqrt{A}\right)^2,$$

and find a function for which equality is achieved.

Problem E.1
Show that both equations (1) $\sin(\tan(x)) = x$ and (2) $\tan(\sin(x)) = x$ have a unique positive solution. Determine which one is greater.

Problem E.2
Find all functions $f\colon [0,\infty) \to [0,\infty)$ such that $2f(3x) + 4f(3y) \leqslant 3f(2x+4y)$ for all $x, y \geqslant 0$

Problem G.1
Let P_1, P_2, P_3 be points on a parabola, and denote the triangle formed by the tangents to the parabola at these points as $\triangle Q_1Q_2Q_3$. Determine the range of values that can take the ratio of the area of the triangle $\triangle P_1P_2P_3$ to the area of the triangle $\triangle Q_1Q_2Q_3$.

Problem G.2
A disk of radius R is covered by m rectangular strips of width 2 and infinite length. Prove that $m \geqslant R$.

Problem P.1
A group of k people play a game with a box containing k balls with their names. The first player draws the ball and wins if his/her name is on the ball. Otherwise, the ball is returned to the box, and the person whose name was on the drawn ball is the next player to draw. The procedure continues until someone pulls out the ball with his/her name on it.
What is the probability of winning for each player?

Problem P.2
Let $n \in \mathbb{N}$, $n > 1$ and $c \geqslant 1$. Suppose that $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ are two families of independent events in a probability space (Ω, Σ, P) such that $B_k \subset A_k$ and $c \cdot P(B_k) \geqslant P(A_k)$ for every $k = 1, \ldots, n$. Show that

$$c \cdot P(B_1 \cup \cdots \cup B_n) \geqslant P(A_1 \cup \cdots \cup A_n).$$